# Stratification modulo $n$

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## Contents

I am very grateful to Nathan Bowler for supplying a crucial little *aperçu* that enables one to dispense with the axiom of choice for pairs in the proof of one of the principal results (corollary 3). And also for helpful critical comments (in particular spotting an embarrassingly large *lacuna* in the proof of lemma 8) that have sharpened up the presentation greatly. By rights he should be a co-author, but he has asserted his moral right to *not* be identified as the (or even merely *an*) author of this work: his reward will be in heavan . . . and all mistakes are mine.

There could be a section on the partial order discovered by Bowler. Indeed it might even make a self-contained paper. It would contain Bowler's construction of permutations universal for various classes. Corollaries in the form of existence of permutation models containing automorphisms. Are binary disjoint unions binary LUBs? arbitrary LUBs? binary infs? We also need to consider the partial order where the embedding function is not injective. What do we lose? We need to say something about how we really are doing this stuff on Symm($V$) not just any old infinite symmetric group.

# 1 Introduction and Summary

Recently Zuhair Abdul Ghafoor Al-Johar [19] has directed our attention to a syntactic constraint that is – on the face of it – *tighter* than NF's device of stratification[1]; in this little essay I consider a *weakening*, namely the generalisation of stratification to *stratification modulo* $n$. So far the coterie of NFistes has considered neither the possibility that the class of unstratified formulæ in the language of set theory might admit any structure or gradation, nor the possibility that failure-of-stratification (which perhaps we can call *dysstratification*) might come in degrees, let alone the possibility that recognition of such degrees might allow one to gain understanding and prove useful facts.

So stratification-mod-$n$ opens a new vein, and the purpose of this note is to advertise some nuggets and prepare the ground for future nuggets. It has to be admitted that stratification-mod-$n$ comes across as a highly artificial notion, of interest only to those whose tastes have been primed by prior exposure to the idea of stratification. However, as we shall see below, there are familiar set-theoretic notions that are stratifiable-mod-$n$ so the concept is not vacuous

[1] Though recent work of Nathan Bowler seems to establish (modulo some very minor set-theoretic assumptions) that every stratifiable formula is equivalent to an acyclic formula. I do not yet understand his proof, and he hasn't published it. However I see no reason to doubt it.

in practice. Further, there is a nontrivial result that makes essential use of this notion, and we will see it in section 8 where we show (theorem 6) that – for NF – duality for formulæ that are stratifiable-mod-2 is consistent relative to NF. Although we do not believe that this result is best possible it is nevertheless worth mentioning beco's it is a significant improvement on what has so far been known about duality. We still believe that duality for *all* formulæ is consistent relative to NF. If we achieve that, stratification-mod-$n$ can perhaps go back to the shades whence it came. But perhaps by then it will have thrown useful light on other ideas: we shall see.

## 2 Stratification

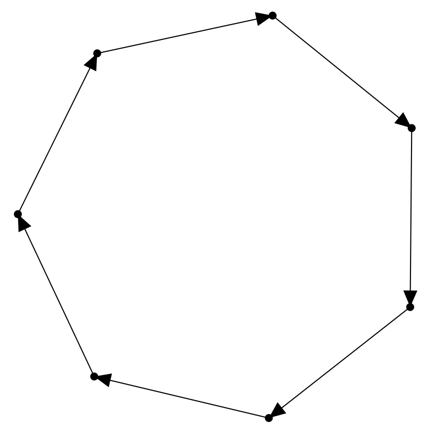
Even readers who are familiar with the idea of stratification should probably read this section, since the treatment here is slightly more abstract than the usual one, and is tailored to the developments that follow.

Let $\mathcal{L} = \mathcal{L}(\in, =)$ be the language of set theory. We associate to every formula $\phi \in \mathcal{L}$ a digraph as follows. First we identify two variables '$v$' and '$v'$' if $\phi$ contains either of the atomic subformulæ '$v = v'$' or '$v' = v$', and so on, recursively. The vertices of the digraph are the equivalence classes of variables in $\phi$, and we place a directed edge from one vertex $[v]$ to another vertex $[v']$ if the atomic formula '$v \in v'$' is a subformula of $\phi$.

We call this graph the *derived graph* of $\phi$, and write it $G_\phi$.

Our digraphs may have multiple edges in the restricted sense that there could be a directed edge from $v$ to $v'$ as well as a directed edge from $v'$ to $v$ – but only one in each direction. In a digraph we can have a special notion of a path from $v_1$ to $v_2$ which allows us to "go the wrong way". The **length** of such a path is computed by adding 1 every time you follow an arrow the right way, and subtracting 1 every time you go the wrong way.

For $n \leq \aleph_0$ the $n$-**gon** $G_n$ is the unique connected digraph with precisely $n$ vertices where every vertex has indegree 1 and outdegree 1. It is a reduct of the integers mod $n$, in that it has successor-mod-$n$ but does not have addition or multiplication. Despite this document bearing the title "stratification mod $n$" arithmetic mod $n$ plays essentially no rôle in what follows: if we are to sensibly describe the circular stratification that is of interest to us here then it is the $n$-gon $G_n$ that we need – rather than $\mathbb{Z}/n\mathbb{Z}$ – because the additive and multiplicative structures of $\mathbb{Z}/n\mathbb{Z}$ do nothing for us when computing stratifications; they are merely distractions.



$G_7$

Unlike the integers-mod-$n$ the $n$-gon $G_n$ is not rigid: its automorphism group is the cyclic group[2] $C_n$. This matters because the set of stratifications-mod-$n$ of a formula $\phi$ are "closed under rotation" so that if there is one there are $n$.

There is a slight problem when $n = 2$, since digraphs cannot normally have multiple edges, but we will tough this one out. And we still entertain hopes that the $\aleph_0$-gon will turn out to have a name already. For the moment let's call it the $\mathbb{Z}$-gon.

The theory of $n$-gons is Horn, so the class of $n$-gons is closed under products and homomorphisms. In particular there is a homomorphism $G_m \twoheadrightarrow G_n$ whenever $n$ divides $m$, and we will exploit this fact, for example in the proof of remark 1.

**Definition 1**
*A* **stratification graph** *is one where*

$$(\forall v_1)(\forall v_2)(\textit{all paths from } v_1 \textit{ to } v_2 \textit{ are the same length}).$$

*A* **stratification-mod-$n$** *graph is one with a homomorphism onto the $n$-gon. If we don't want to mention the '$n$' we will say that a graph that is stratified-mod-$n$ is* **circularly stratified***.*
*Equivalently a graph is a stratification-mod-$n$ graph iff, for any two vertices $v_1$ and $v_2$, all paths from $v_1$ to $v_2$ have the same length modulo $n$.*

This definition doesn't make a great deal of sense unless $n \geq 2$, and we will only use it in those settings.

**Definition 2**
*A formula is* **stratifiable** *iff its derived digraph is a stratification graph.*

[2]Not $D_n$! Dana Scott points out that we can't think of $G_n$ as a mere polygon because otherwise reflections would be automorphisms, and they aren't, so we have to think of them as polygons with directed edges.

*A* **stratification** *of a formula $\phi$ is a homomorphism from the derived graph $G_\phi$ of $\phi$ to the $\mathbb{Z}$-gon;*

*A* **stratification-mod**-*$n$ of a formula $\phi$ is a homomorphism from the derived graph $G_\phi$ of $\phi$ onto the $n$-gon.*

*A formula is* **stratifiable mod** *$n$ iff its derived digraph is a stratification-mod-$n$ graph.*

*Again, if we do not want to mention the 'n' we will say of a formula that is stratifiable-mod-n that it is* **circularly stratifiable**.

Equivalently a stratification graph is one where, for all vertices $v$, all paths from $v$ to $v$ are of length 0; a stratification-mod-$n$ graph is one where, for all vertices $v$ and $v'$, all paths from $v$ to $v'$ are of the same length mod $n$, or – equivalently – for all vertices $v$, all paths from $v$ to $v$ are of length 0 mod $n$.

Observe that, for each $n$, the theory of stratification-mod-$n$ graphs is a first-order theory, indeed a universal theory.

A **moiety** is a set $x$ such that $|x| = |V| = |V \setminus x|$.

A formula is **(Crabbé)-elementary** iff all its variables are related by the ancestral of the relation "$v$ and $v'$ occur in an atomic subformula together". We will often tacitly assume in what follows that all our formulæ are Crabbé-elementary. Classically (though not constructively) every first-order formula is equivalent to a boolean combination of elementary formulæ (and every *closed* first-order formula is equivalent to a boolean combination of *closed* elementary formulæ) so there is little cost in making this simplifying assumption. Without it, some of the proofs below would become snarled up in annoying minor details, so we plead for the reader's indulgence.

**Remark 1**

*(i) A formula that can be stratified both mod-n and mod-m can be stratified mod-LCM($m, n$), and conversely.*

*(ii) A formula that is stratifiable-mod-n for arbitrarily large $n$ is just plain stratifiable, and a stratifiable formula is stratifiable-mod-n for all $n$.*

*Proof:*

(i) Let $\phi$ be such a formula, and $G_\phi$ its derived graph. $\phi$ is both stratifiable-mod-$n$ and stratifiable-mod-$m$ which is to say that there are homomorphisms $f : G_\phi \twoheadrightarrow G_n$ and $g : G_\phi \twoheadrightarrow G_m$. Consider the graph $G = \{\langle f(v), g(v)\rangle : v \in G_\phi\}$ with the obvious edge relation. We want to show that $G$ is the LCM($m, n$)-gon. It is a graph of size at most $n \cdot m$. There is a homomorphism $\lambda v.\langle f(v), g(v)\rangle : G_\phi \twoheadrightarrow G$. Clearly every vertex in $G$ has indegree 1 and outdegree 1, so it is either a gon (if it is connected) or a union of gons (o/w). It is also clear that if we apply the edge operation of the graph $G$ $n$ times to an ordered pair we reach an ordered pair with the same first component, and if we apply the edge operation $m$ times to an ordered pair we reach an ordered pair with the same second component, so if we apply the edge operation LCM($m, n$) times to an ordered pair we get back to that same ordered pair. And LCM($m, n$) is the smallest number of times we can apply the edge operation of $G$ to secure this effect. Therefore one of the connected components of $G$ is the LCM$(m,n)$-gon, so $G$ is the LCM$(m,n)$-gon as long as it is connected.

To establish that it is – indeed – connected, we show that, for all vertices $v, v'$ in $G$, there is a path from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$. Recall that $G_\phi$ is a stratification graph, so there is a well-defined distance, $d$, from $v$ to $v'$. We can now see that the distance from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$ is precisely $d$, so $G$ is connected.

For the converse, if $\phi$ is stratifiable-mod-LCM$(m,n)$ then there is a homomorphism $f : G_\phi \twoheadrightarrow G_{LCM(m,n)}$. We compose $f$ with the homomorphism from $G_{LCM(m,n)}$ onto $G_n$, thereby showing that $\phi$ is stratifiable-mod-$n$; similarly $\phi$ is also stratifiable-mod-$m$.

(ii) If $n >$ length$(\phi)$, then any stratification-mod-$n$ of $\phi$ is (or, more correctly, can be easily modified into) a stratification. For the other direction, observe that, for every $n$, the $\mathbb{Z}$-gon maps onto the $n$-gon $G_n$. ■

Assertion (ii) is literally true, but the reader should be warned not to misread it as "any expression that, for each $n$, is *equivalent to* a formula that is stratifiable-mod-$n$ is equivalent to a stratified formula"; in section 3 we will see counterexamples to this stronger assertion.

So the picture is: we only have to worry about stratifiability-mod-$p$ for $p$ prime, and the various stratifiabilities-mod-$p$ are the weakest conditions; stratifiability-mod-$m \cdot n$ is a weaker condition than stratifiability-mod-$n$, and all these are weaker than stratifiability *tout court*, which is the conjunction of them all. The various stratifiabilities-mod-$p$ with $p$ prime all seem to be equally weak, and they are all of minimal strength.

It may be worth noting that we cannot strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* both to a formula that is stratifiable-mod-$n$ and to a formula that is stratifiable-mod-$m$, because of the Axiom of Counting. For every $n$, the Axiom of Counting is equivalent (modulo NF) to a formula that is stratifiable mod $n$ (we will see a proof of this on p 22) so the analogue of remark 1 part (ii) would tell us that it is equivalent to a stratifiable formula. However, it is known that it is not equivalent (modulo NF) to any stratifiable formula. However, the Axiom of Counting is invariant, so it might be possible to strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* (mod NF) both to a formula that is stratifiable-mod-$n$ and to a formula that is stratifiable-mod-$m$, if the conclusion we want to infer is that the formula in question is merely invariant (modulo NF) rather than actually stratifiable. Explain 'invariant'?

## 2.1 Wrapping up miscellaneous definitions

Finally we wrap up some definitions and notations. Some of them are standard in an NF context but a clear summary of them can do no harm.

- We write '`RUSC`$(R)$' for the relation that some might write as '$R^\iota$', namely $\{\langle\{x\},\{y\}\rangle : \langle x, y\rangle \in R\}$. This $R$ was binary, but we can define `RUSC` for relations of higher arity analogously. (The acronym is from [?] Relation Unit Sub Class

- $\iota$ is the singleton function: $\iota(x) = \{x\}$.
  If $\iota \restriction x$ exists we say $x$ is **strongly cantorian**.
- We write 'Symm$(X)$ for the full symmetric group on $X$.
  In practice $X$ is always $V$.
- $j : \text{Symm}(V) \to \text{Symm}(V)$ is defined so that $j(\sigma)(x) = \sigma``x$.
- Let us use lower-case $\mathfrak{fraktur}$ characters for variables ranging over conjugacy classes.
- If there is $\sigma$ such that $(j\sigma)^{-1} \cdot \tau \cdot \sigma = \pi$ then we say $\tau$ and $\pi$ are **skew-conjugate**. Observe that this relation of skew-conjugacy is in fact an equivalence relation. However the definition is not stratified and (in NF) the graph of the relation is not a set and the equivalence classes are not sets. The skew-conjugacy class of $\mathbb{1}$, the identity relation, is the class of internal automorphisms and it should be easy to show that that need not be a set (though I have not done so so far!)

The significance of this relation is that skew-conjugate permutations give rise to isomorphic permutation models, as follows. Suppose $\tau$ and $\pi$ are skew-conjugate; then $x \in \tau(y)$ iff $x \in (j\sigma)^{-1} \cdot \pi \cdot \sigma(y)$ iff $\sigma(x) \in \pi \cdot \sigma(y)$ which is as much as to say that $\sigma$ is an $\in$-isomorphism between $V^\tau$ and $V^\pi$.

However skew-conjugacy doesn't seem to be a congruence relation for very much. Certainly not for the group-theoretic operations of product or inverse.

Every permutation is conjugate to its inverse, but examples can be found of permutations not skew-conjugate to their inverses. For example, if $\tau$ is the 3-cycle $(\emptyset, \{\emptyset\}, \{\{\emptyset\}\})$ then, in $V^\tau$, $\emptyset$ has become a Quine atom. However in $V^{\tau^{-1}}$ there will be no Quine atom (unless there was one in $V$) so $V^{\tau^{-1}}$ and $V^\tau$ are not isomorphic and $\tau$ is not skew-conjugate to its inverse.

**Minding your $p$s and $q$s:**

A **partition** $\mathbb{P}$ of a set $X$ is a set of pairwise disjoint subsets of $X$ s.t. $\bigcup \mathbb{P} = X$. The members of $\mathbb{P}$ are **pieces** (of $\mathbb{P}$). I shall use the letter '$\mathbb{P}$' to range over partitions.

'$\Pi$' will be used for **products** (in particular for products of (pairwise disjoint) transpositions, as in '$\Pi_{x \in A}(x, V \setminus x)$').

$\mathcal{P}(x)$ is the **power set** of $x$; $B(x) = \{y : x \in y\}$, the principal ultrafilter in the powerset algebra $\mathcal{P}(V)$ generated by $\{x\}$; $\rotatebox[origin=c]{180}{$\mathcal{P}$}(x)$ is $\{y : y \cap x \neq \emptyset\}$; to put it another way it is thus dual to $\mathcal{P}$, in the sense that $\rotatebox[origin=c]{180}{$\mathcal{P}$}(x) = V \setminus (\mathcal{P}(V \setminus x))$ – which is why we write it with an upside-down '$\mathcal{P}$'. The fact that $B(x) = \rotatebox[origin=c]{180}{$\mathcal{P}$}(\{x\})$ also helps.

# 3 Motivating stratification-mod-$n$

It's already quite a challenge to persuade set theorists that ordinary mere stratification is a natural and well-motivated notion, tho' they are open to persuasion and a case can be made: stratification sits very well with the endogenous strong typing of mathematics. This can be seen for example, in the type-inspired treatment of Burali-Forti (see [7] for example). Stratification-mod-$n$ is much less natural so motivating it is an altogether taller order; however there is a story that can be told ....

### 3.1 The $\in$-game

The $\in$-game $G_x$ in [13] is played by two players – `I` and `II`– and is initiated by player `I` picking a member of $x$; thereafter the players move alternately, each picking an element of the other's previous choice until one of them attempts to pick a member of the empty set and thereby loses. (That is the only way the game can end). This subject matter has a naturally stratifiable-mod-2 flavour: "Player `I` has a Winning strategy in $G_x$" and "Player `II` has a Winning strategy in $G_x$" are both stratifiable-mod-2. The first is

$$(\forall y)(\text{Ƅ}(\mathcal{P}(y)) \subseteq y \to x \in y)$$

which is as much as to say that $x$ belongs to the $\subseteq$-least fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(X))$. The second is

$$(\forall y)(\mathcal{P}(\text{Ƅ}(y)) \subseteq y \to x \in y).$$

which is as much as to say that $x$ belongs to the $\subseteq$-least fixed point for $X \mapsto \mathcal{P}(\text{Ƅ}(X)))$.

### 3.2 The Axiom of Counting

The axiom of counting is unstratified and not equivalent modulo NF to any stratifiable formula but is, for each concrete $n$, equivalent modulo NF to a formula that is stratifiable-mod-$n$. It's also invariant. The same goes for $\text{AxCount}_{\leq}$ (with a bit more work) since – for any concrete $k$ – $\text{AxCount}_{\leq}$ can be written as '$(\forall n \in \mathbb{N})(n \leq T^k n)$'.

See also section 5.

## 4 Preservation Results for Stratification-mod-$n$

We start with a definition from [6].

**Definition 3** $H(0, \tau) =: \mathbb{1}_V$; $\quad H(n+1, \tau) =: (j^n \tau) \cdot H(n, \tau)$.

This $H$ notation will only ever be used with concrete naturals in first argument place.[3]

[3] so we shouldn't use these purely concrete things as arguments; they should be hidden in the syntax? The trouble with this policy is that we don't want footnotesized things like '$LCM(n, m)$'.

The effect of this notation is that, for any $\tau$ and any concrete $n$, $(\forall xy)(x \in \tau(y) \longleftrightarrow H(n,\tau)(x) \in H(n+1,\tau)(y))$. The intention behind the design of this family of permutations derived from a single $\tau$ is to prove that, when $\phi$ is stratifiable, $\phi^\tau$ is equivalent to the result of replacing every occurrence of each free variable '$v$' with '$H(n_v,\tau)(v)$' where $n_v$ is the concrete natural number associated to the variable '$v$' in a fixed stratification of $\phi$. In the treatment here, our stratifications are functions from $\mathit{vbls}(\phi)$ to the $\mathbb{Z}$-gon or the $n$-gon and do not take numbers as values. This can be remedied by composing a stratification with a decoration-by-numbers (satisfying the obvious adjacency condition) of the gon in question.

It might be worth minuting other facts about the family of permutations engendered in this way from a permutation $\sigma$. For example $H(n+m,\sigma) = j^m(H(n,\sigma)) \cdot H(m,\sigma)$. We don't think there is a nice formula for $H(n \cdot m,\sigma)$. This is another manifestation of the fact that there is no natural arithmétic structure on the set of type indices.

We have a theorem of Scott that stratifiable formulæ are preserved under the Rieger-Bernays permutation construction. This is an assertion of the form

Might be an idea to cite it

$$(\forall\pi)(F(\pi) \to (\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi))) \tag{A}$$

or equivalently

$$(\forall\phi)(\phi \in \Gamma \to (\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)))$$

Assertions like (A) have converses of the form

$$(\forall\pi)[(\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)] \tag{B}$$

and of the form

$$(\forall\phi)[(\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma] \tag{C}$$

In this section we consider the project of proving assertions like these where $\Gamma$ is the set of formulæ that are stratifiable-mod-$n$. This will involve us in identifying interesting properties of permutations to serve as the '$F$' in the statement of the results

## 4.1 Instances of (A): $(\forall\pi)(F(\pi) \to (\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)))$

**Proposition 1** *If $\phi$ is stratifiable-mod-$n$ then it is preserved under all Rieger-Bernays constructions using setlike permutations $\pi$ s.t. $H(n,\pi) = \mathbb{1}$.*

*Proof:*

The proof is a straightforward adaptation of the proof given by Henson.[**?**] Theorem 1.2 and corollary 1,4.

In Henson's treatment of the stratified case we fix a stratification $s$ for $\phi$. [In that treatment stratifications take values in $\mathbb{Z}$, not in the $\mathbb{Z}$-gon.] Then, whenever we look at a subformula '$x \in \sigma(y)$' in $\phi^\sigma$ we replace it by '$H(n,\sigma)(x) \in H(n+1,\sigma)(y)$' where $n$ is the type given to the variable '$x$' by the stratification $s$. We then observe that, for every variable, all occurrences of that variable in the rewritten version of $\phi^\sigma$ are prefixed by a '$H(n,\sigma)$' where $n$ is the type given to '$x$' by the stratification $s$. Then we appeal to the fact that $H(n,\sigma)$ is a permutation, so we can reletter '$H(n,\sigma)(x)$' as '$x$', and this manipulation turns $\phi^\sigma$ back into $\phi$. The difference here, in this case, is that our subscripts are no longer integers but are integers-mod-$n$, so that if $i \equiv j \pmod n$ we must have $H(i,\sigma) = H(j,\sigma)$. This is equivalent to requiring that $H(n,\sigma)$ be the identity. ■

## 4.2 Instances of (C): $(\forall\phi)[(\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma]$

There is a theorem, proved by Pétry and Forster ([9], [17], [18]) to the effect that: if a formula is preserved under all Rieger-Bernays constructions using setlike permutations then it is equivalent to a stratifiable formula.

Is there an analogous result to the effect that if a formula is preserved under all Rieger-Bernays constructions using setlike permutations $\sigma = H(n,\sigma)$ then it is equivalent to a formula that is stratifiable-mod-$n$? Something like that ought to be true, and it's probably worth proving. Put up or shut up

## 4.3 Instances of (B): $(\forall\pi)[(\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)]$

We start with a very easy example:

**Remark 2** *If $f : V \to V$ (possibly a proper class) satisfies $\phi \longleftrightarrow \phi^f$ for all stratifiable expressions then $f$ must be a setlike permutation.*

Setlike?

*Proof:* The axiom of extensionality is stratifiable, and any $f$ that preserves it must be onto. If $f$ preserves an $(n+1)$-stratifiable formula then $H(n,f)$ has to be defined, so $f$ has to be $n$-setlike. ■

One might expect that if $\pi$ is a permutation that preserves all formulæ that are stratifiable-mod-$n$ then $H(n,\pi) = \mathbb{1}$. Something with that sort of flavour should be true. The following is a straw in the wind.

**Remark 3** *If $H(n,\sigma) = \mathbb{1}$ and $H(k,\sigma) = \mathbb{1}$ then $H(HCF(n,k),\sigma) = \mathbb{1}$.*

*Proof:*

This is because, for every $\sigma$, the class of naturals $n$ s.t. $H(n,\sigma) = \mathbb{1}$ is closed under subtraction[4] so we can, as it were, perform Euclid's algorithm. If $H(n,\sigma) = \mathbb{1}$ and $H(k,\sigma) = \mathbb{1}$, with $n > k$ then reflect that $H(n,\sigma)$ is $(j^k H(n-k,\sigma)) \cdot H(k,\sigma)$. So $j^k H(n-k,\sigma) = H(n,\sigma) \cdot H(k,\sigma)^{-1} = \mathbb{1} \cdot \mathbb{1} = \mathbb{1}$. But then $H(n-k,\sigma) = \mathbb{1}$ as well. ■

[4] *prima facie* we cannot expect this thing to be a set, since it is defined by an unstratified expression.

This doesn't actually say that if $\sigma$ both preserves formulæ that are stratifiable-mod-$n$ and preserves preserves formulæ that are stratifiable-mod-$k$) then it preserves formulæ that are stratifiable-mod-$HCF(n, k)$, but it has that flavour.

One wants to say that a permutation that preserves *all* closed formulæ must be an $\in$-automorphism, but that doesn't seem to be strictly true. At any rate we don't know how to prove it! Perhaps we can prove it by reasoning about Ehrenfeucht games. What we *do* know how to prove is that, if $V \simeq V^\sigma$, then $\sigma$ is skew-conjugate to the identity. The only permutation that preserves *all* expressions (i.e., including open formulæ) is $\mathbb{1}$.

And, once we have identified predicates $F$ that appear in theorems of flavour (B), one wants to find a structure for the set of all permutations on $V$ such that, for each $F$, the class of permutations that are $F$ is a sub*structure* not a mere sub*class*.

One thing one might hope to prove is that if $\phi$ is stratifiable-mod-$n$ and is logically equivalent to a formula that is stratifiable-mod-$m$ then it is logically equivalent to a formula that is stratifiable-mod-$nm$ ...

Thinking aloud about this... Editing needed here!!

If $\phi$ is equivalent to something that is stratifiable-mod-$n$ then $\phi$ is equivalent to $(\forall\sigma)(H(n, \sigma) = \mathbb{1} \to \phi^\sigma)$. Now let$\tau$ be a permutation satisfying $H(m, \tau) = \mathbb{1}$ and consider what happens in $V^\tau$. We have

$(\forall\sigma)(H(n, \sigma) = \mathbb{1} \to \phi^\sigma)^\tau$

This requires thought! "$H(n, \sigma) = \mathbb{1}$" can be thought of as a stratified formula with $n$ occurrences of '$\sigma$', one at each of $n$ distinct adjacent types, namely

$$j^n(\sigma) \cdot j^{n-1}(\sigma) \cdot \cdots \cdot j^{n-2}(\sigma) \cdot \sigma = \mathbb{1}$$

$$j^n(H(k, \tau)(\sigma)) \cdot j^{n-1}(H(k{+}1, \tau)(\sigma)) \cdot j^{n-2}(H(k{+}2, \tau)(\sigma)) \cdots H(k{+}n, \tau)(\sigma) = \mathbb{1}$$

where we have chosen $k$ large enough so that $H(k + 1, \tau)(\sigma)$ is conjugated to $H(k, \tau)(\sigma)$ by $\tau$ or $j^i(\tau)$

Now let's think about what happens to $\phi^\sigma$ in the permutation model. This is a problem well-known to your humble correspondent.

$(x \in \sigma(y))^\tau$ becomes

$H(k - 1, \tau)(x) \in H(k + 1, \tau)(\sigma)(H(k, \tau)(y))$ becomes

$x \in (\tau_k)^{-1}$“$\tau_{k+2}(\sigma)(\tau_{k+1}(y))$

Need to continue rewriting. Editing needed here!!

Now we reletter '$\tau_k(\sigma)$' as '$\sigma$' throughout.

$j^n(\tau_k(\sigma)) \cdot j^{n-1}(\tau_{k+1}(\sigma)) \cdot \cdot j^{n-2}(\tau_{k+2}(\sigma)) \cdots \tau_{k+n}(\sigma) = \mathbb{1}$ becomes

$$j^n(\sigma) \cdot j^{n-1}(\sigma^\tau) \cdot \cdot j^{n-2}(\sigma^{\tau_2} \cdots \sigma^{\tau_n}) = \mathbb{1}$$

which is

$$j^n(\sigma) \cdot j^{n-1}\tau j^{n-1} \cdot \sigma \cdot j^{n-1}\tau^{-1}) \cdot \cdot j^{n-2}(\sigma^{\tau_2}) \cdots \sigma^{\tau_n} = \mathbb{1}$$

and we can do some cancellation...

Definitely work to be done in section 4; not sure what was going on there(!) Editing needed here!!

## 5 Stratifiable-mod-$n$ for every $n$

Given a theory $T$, there is a natural class consisting of those formulæ that, for each $n$, are equivalent modulo $T$ to a formula that is stratifiable-mod-$n$. It is larger than the class of stratifiable formulæ and even (though this is less obvious and of course depends on $T$) contains formulæ that are not $T$-invariant. Whether or not there are formulæ that are $T$-invariant but are not in our class i do not know at this stage.

We will consider the following formulæ: stcan$(x)$, WF$(x)$, $\bigcup x \subseteq x$.

As we shall see, the axioms TCl and TCo of transitive closure and transitive containment are both of this class.

We need an adjective for formulæ that, for each $n$ are $T$-equivalent to a formula that is stratifiable-mod-$n$!

**Definition 4**
*Let us say $x$ is $n$-**hemitransitive** iff $(\forall y)(y \in^{n+1} x \to y \in x)$.*

Thus ordinary transitivity is 1-hemitransitivity. It is easy to see that $n$-hemitransitivity is stratifiable-mod-$n$. It is also easy to see that transitivity implies $n$-hemitransitivity for all $n$.

We know that the property of being a transitive set cannot be captured by a stratifiable formula if NF is consistent (We know that the collection of transitive sets is a proper class). Presumably the property of being $n$-hemitransitive cannot be captured by any stratifiable formula either – and for the same reasons – but I haven't written out the details and i don't think anyone else has either. This is worth proving

Like transitivity, $n$-hemitransitivity is a Horn property, so there is a notion of the $n$-hemitransitive closure of a set. Observe that if $x$ is $n$-hemitransitive then $x \cup \bigcup x \cup \ldots \bigcup^{n-1} x$ is transitive. So "There is an infinite transitive set" is, for each $n$, equivalent to "There is an infinite $n$-hemitransitive set" …which of course is stratifiable-mod-$n$. So it ought to be invariant. Suppose then that we are working in a model of NF + AxCount$_\leq$ that contains $V_\omega$. Holmes' clever permutation will kill off $V_\omega$ but it would leave behind an infinite transitive wellfounded set. This doesn't seem frightfully plausible. Tidy this up

Now we can see that the fact that if $x$ is $n$-hemitransitive then $\bigcup^{<n} x$ is transitive means that the axiom of transitive containment belongs to our special class of formulæ.

Let TCl$_n$ say that every set has an $n$-hemitransitive closure. Consideration of $\bigcup^{<n} X$ shows that this implies TCl. For the other direction we will need unstratified separation – annoyingly. TCo$_n$ implies TCo; for the other direction we do not need any unstratified separation, since any transitive set is $n$-hemitransitive for any $n$.

Consider the sets – call them $X_n$ for the nonce – where $X_n$ is the least fixpoint for $x \mapsto (\mathcal{P}_{\aleph_0})^n(x)$. $X_n$ is the collection of sets of (finite) rank a multiple of $n$. The assertion that $X_n$ exists is stratifiable-mod-$n$, and $V_\omega$ is of course $\bigcup_{n \in \mathbb{N}} X_n$. Thus the assertion that $V_\omega$ exists is, for each $n$, equivalent (over NF) to a formula that is stratifiable-mod-$n$. However, beco's of Holmes' clever permutation, it is not invariant!

This shows that, for $T = \mathrm{NF} + \mathrm{AxCount}_\leq$ at least, there are formulæ that, for each $n$, are $T$-equivalent to something that is stratifiable-mod-$n$ but are not $T$-invariant. This doesn't prove the same for NF, but the damage is done. Can we do the same for NF? Clearly one wants to put Holmes' clever permutation to use. Let $A_n$ be the assertion that there is an infinite $n$-hemitransitive subset of $X_n$. Is $A_n$ equivalent to the assertion that there is an infinite transitive subset of $V_\omega$? I can only see the implication one way.

More work needed here

Probably the most important unstratified set-theoretic property is wellfoundedness; it cannot be captured by any stratifiable formula, but can it be captured by a formula that is stratifiable-mod-$n$? The following elementary observation took us by surprise.

**PROPOSITION 2** *"$x$ is wellfounded" is, for every $n$, equivalent to a formula that is stratifiable-mod-$n$.*

*Proof:*

In fact there is a parametrised family of such formulæ. The typical formula is $(\forall y)(\mathcal{P}^n(y) \subseteq y \to x \in y)$, or $\mathrm{WF}_n(x)$ for short. Notice that for $n = 1$ this gives the natural inductive definition of the class of wellfounded sets as the least fixpoint for the power set function. $\mathrm{WF}_n(x)$ is stratifiable-mod-$n$ all right, but is it equivalent to $WF(x)$? One direction is easy: $\mathcal{P}^n(y) \subseteq y$ is a weaker condition than $\mathcal{P}(y) \subseteq y$ so if you belong to everything satisfying the weaker condition you certainly belong to everything satisfying the stronger condition. So $\mathrm{WF}_n(x)$ implies $\mathrm{WF}(x)$. What about the other direction?

Let us say that a set $y$ s.t. $\mathcal{P}^n(y) \subseteq y$ is $n$-**fat**[5]. Observe that if $y$ is $n$-fat so is $\mathcal{P}(y)$. Suppose now (with a view to performing an $\in$-induction) that every member of $x$ belongs to every $n$-fat set. Then $x$ is included in every $n$-fat set, and so is a member of the power set of any $n$-fat set. So it is a member of every $n$-fat set. Thus we can prove by $\in$-induction that every wellfounded set is $\mathrm{WF}_n$. ■

"$(\forall y)(\mathcal{P}^n(y) \subseteq y \to x \in y)$" seems to make sense only in NF-like contexts, where separation fails and sets can be supersets of their own power sets. However if we contrapose and replace '$y$' by '$V \setminus y$' we obtain

$$(\forall y)(x \in y \to (\exists z \in y)(z \notin \mathcal{P}^n(V \setminus y))).$$

which make sense in a context with full separation. This development is analogous to the way in which one obtains the concept of *regular set* from the natural

[5]This terminology is generalised from that in [2].

inductive (least-fixpoint) definition of wellfounded set as $(\forall y)(\mathcal{P}(y) \subseteq y \to x \in y)$.

While we are about it (though perhaps this observation could be better placed elsewhere) this shows that although stratified parameter-free $\in$-induction seems to be quite weak (it is open whether or not it proves anything more than the nonexistence of a universal set) it is nevertheless the case that, for each $n$, parameter-free $\in$-induction for formulæ that are stratifiable-mod-$n$ implies full $\in$-induction.

Editing needed here!!

## 5.1 Von Neumann Ordinals

So $x$ being a wellfounded set can be captured, for any $n$, by a formula that is stratifiable-mod-$n$. What about being a von Neumann ordinal?

Let's stick to the case $n = 2$ for the moment. How about we say $\alpha$ is a von Neumann ordinal iff it is a (wellfounded) $n$-hemitransitive set of $n$-hemitransitive sets. How's that for a definition of $\alpha$ being a von Neumann ordinal? $\alpha$ is a wellfounded hemitransitive set of hemitransitive sets (by analogy with a wellfounded transitive set of transitive sets). The trouble with this is that $\{1, 3\}$ satisfies these criteria and is not an ordinal. We could add the further condition that $(\forall xy \in \alpha)(x \subseteq y \vee y \subseteq x)$ (which is satisfied by von Neumann ordinals and stratifiable-mod-2). Although that doesn't dispose of $\{1, 3\}$ it may hold the key. Add the condition that $\alpha$ should belong to a hemitransitive set $X$ satisfying $(\forall xy \in X)(x \subseteq y \vee y \subseteq x)$. Observe that any hemitransitive set containing $\{1, 3\}$ must also contain 2, but $2 \not\subseteq \{1, 3\}$ and $\{1, 3\} \not\subseteq 2$.

Thus we have the following **stratifiable-mod**-2 definition of von Neumann ordinal:

A set $\alpha$ is a von Neumann ordinal iff

(i) It is a wellfounded 2-hemitransitive set of 2-hemitransitive sets;

(ii) it belongs to a 2-hemitransitive set $Y$ satisfying $(\forall xy \in Y)(x \subseteq y \vee y \subseteq x)$.

Let us prove that this definition succeeds. First thing is to check that the definition as (wellfounded) 2-hemitransitive set of 2-hemitransitive sets *very nearly* powers an $\in$-induction to show that every 2-hemitransitive set of 2-hemitransitive sets is an ordinal. For the induction we want that every 2-hemitransitive set of von Neumann ordinals is a von Neumann ordinal. If $X$ is a 2-hemitransitive set of ordinals then it contains any ordinal $\alpha$ such that $(\exists \beta \in X)(\alpha + 1 < \beta)$. Thus, if the set of ordinals in $X$ has no largest element than $X$ is an ordinal as desired. If not, you can end up with things of the form $\alpha \cup \{\alpha + 2\}$. But any 2-hemitransitive set containing $\alpha \cup \{\alpha + 2\}$ will contain $\alpha + 1$ and violate the $\subseteq$-connexity condition.

To complete the proof for the case where $\alpha$ genuinely is an ordinal observe that $\alpha \cup \{\alpha\}$ is a 2-hemitransitive set containing $\alpha$ and satisfying the $\in$-connexity condition.

■

Thus we have proved that

**Proposition 3** *"$x$ is a von Neumann ordinal" is equivalent to a formula that is stratifiable-mod-2.*

It remains to see whether or not one can give stratifiable-mod-$n$ definitions of von Neumann ordinal for other $n$. Even the mod-2 definition we have is unsatisfactory in that the existential quantifier in clause (ii) prevents it being $\Delta_0$.

Editing needed here!!

Here's one that might work. $x$ is a von Neumann ordinal if it is hereditarily $n$-hemitransitive and $(\forall yz \in x)(z \in \bigcup^{n-1} y \vee y \in \bigcup^{n-1} z)$.

## 6 Cylindrical Types

Stratifiable formulæ of the language of set theory are those from which one can obtain wffs of TST by decorating the variables with indices indicating what levels they belong to. There is an analogous move to be made with formulæ that are stratifiable-mod-$n$: one can obtain from them formulæ that are wffs of a typed theory of sets whose levels are indexed by $n$-gons. The more properties we succeed in capturing with formulæ that are stratifiable-mod-$n$ the greater the expressive power of these typed theories of sets will be.

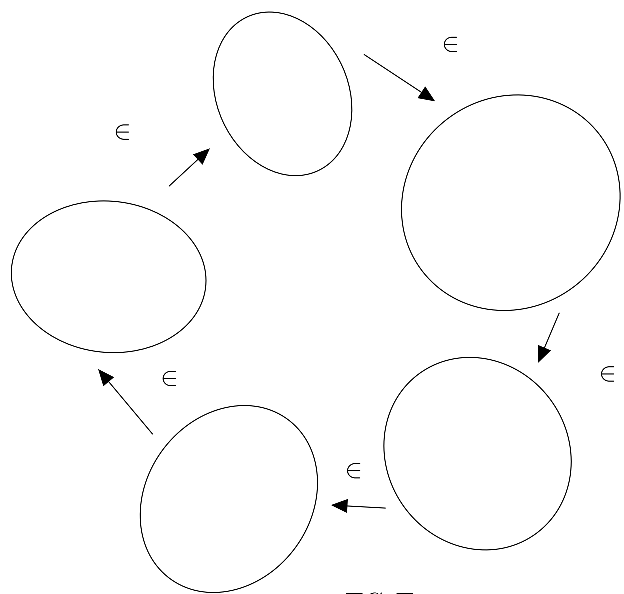
We should note that – in contrast to stratification *tout court* – stratification-mod-$n$ is not a useful notion from the point of view of comprehension principles in a one-sorted language, since there are paradoxical objects that are the extension of formulæ that are stratifiable-mod-$n$; one thinks of the $n$-fold Russell class $\{x : x \not\in^n x\}$ – being the extension of the formula '$x \not\in^n x$' (which is stratifiable-mod-$n$) which is a paradoxical object even in mere first-order logic. This is discussed in section 4 of [5] and also below). Also, as we have just shown (proposition 2), wellfoundedness is capturable by a formula that is stratifiable-mod-$n$ for any $n$ (and is therefore expressible in $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$). Of course there are no known paradoxical objects defined by stratifiable set abstracts.

Define T$\mathbb{Z}$T somewhere

So *that*'s a dead end, but there is an obvious link from formulæ that are stratifiable-mod-$n$ to the theory T$\mathbb{Z}$T+ Amb$^n$. The usual Specker equiconsistency analysis leads one thence to type theories whose levels are indexed by the $n$-gon. One could perhaps call these theories "type theory mod $n$", and that is what I shall do here; the proper name will be "TC$_n$T" ("theory of $n$ cylindrical types").

Let's be formal about it.

**Definition 5**
*The language $\mathcal{L}(TC_nT)$, where $n$ is a concrete natural number, has two binary relation symbols: '$=$' and '$\in$'.*
*Its variables each have a sort index as an integral part, and those sort indices are precisely the elements of the $n$-gon.*




$TC_5T$

The axioms of $\mathrm{TC}_n\mathrm{T}$ are extensionality at each type, as with $\mathrm{T}\mathbb{Z}\mathrm{T}$, but there is a subtlety with the set comprehension axioms. For obvious reasons one cannot allow $(\exists x)(\forall y)(y \in x \longleftrightarrow y \not\in^n y)$ to be an axiom even though this formula is a wff of $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$ and has the syntactic form of a comprehension axiom, and '$y \not\in^n y$' is a wff of the language. One allows set comprehension only for the old $\mathrm{T}\mathbb{Z}\mathrm{T}$ axioms. To be formal about it, a wff that looks like a comprehension axiom is adopted as an axiom only if it is possible to rejig the type indices in it so that the resulting formula is an axiom of $\mathrm{T}\mathbb{Z}\mathrm{T}$.

Thus the axioms of $\mathrm{TC}_n\mathrm{T}$ are "closed under rotation", or *ambiguous* in traditional parlance.

It may be worth noting that $\mathrm{TC}_n\mathrm{T}$ can expressed as a theory in the usual one sorted first-order language $\mathcal{L}(\in, =)$ of set theory. However, since we will not be making any use of this fact, we feel under no obligation to provide a proof.

Could say a *wee* bit more...

The various analogues of Russell's paradox prevent us from adopting as our comprehension scheme for $\mathrm{TC}_n\mathrm{T}$ the obvious scheme of all expressions of the form $(\forall \vec{x})(\exists y)(\forall z)(z \in y \longleftrightarrow \phi(\vec{x}, z))$ that belong to $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$. Of course the mere fact that the existence of $\{x : x \not\in^n x\}$ is not a comprehension axiom does not *ipso facto* mean that the sets $\{x^i : x^i \not\in^n x^i\}$ cannot exist at any of the $n$ levels, though it will be shown below that first-order logic by itself suffices to show that they cannot exist at all levels simultaneously. $\mathrm{TC}_n\mathrm{T}$ has comprehension axioms and can prove that they cannot exist at even one level. This fact is probably worth minuting.

**Remark 4** *$TC_nT \vdash R^i_n = \{x^i : x^i \not\in^n x^i\}$ does not exist for any $i \leq n$.*

*Proof:*

Reasoning in $\mathrm{TC}_n\mathrm{T}$ we pick on any level $i$ and consider the possibility of the existence of $R^i_n = \{x^i : x^i \not\in^n x^i\}$. Consider $\iota^{n-1}(R^i_n)$; is it a member of $R^i_n$ or not? If it is, then it belongs to an $\in$-loop of circumference $n$, so it is barred from membership of $R^i_n$. So it isn't a member of $R^i_n$. So there are $x_1 \ldots x_{n-1}$ with $\iota^{n-1}(R_n) \in x_1 \in x_2 \in \ldots \iota^{n-1}(R_n)$ an $\in$-loop of circumference $n$. But then $x_1 = R_n$ (peel off the brackets) showing that $R_n \in R_n$ after all. ■

Notice that we have not used very much comprehension. All we have used is the assumption that every set has a singleton. In fact all we need is that for every $x$ there is a nonempty subset of $\mathcal{P}(x)$, the point being that every set of loop-free sets is itself loop-free.

Readers might like to note the curiosity that inside first-order logic pure and simple (without using any set theory at all) we can show that $\{x^i : x^i \not\in^n x^i\}$ must fail to exist at least one level $i$. This, too, is probably worth minuting.

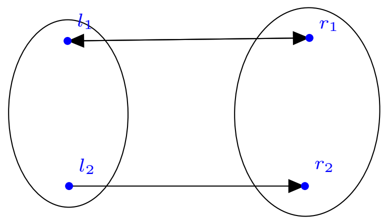
**Remark 5** *It is a theorem of First Order Logic that that no model of $TC_nT$ can contain $R^i_n = \{x^i : x^i \not\in^n x^i\}$ for all $i \leq n$.*

*Proof:*

Suppose $\{x : x \not\in^n x\}$ exists at every level. Let us write '$R^i$' for its manifestation at level $i$. Let $i$ be an arbitrary concrete natural $\leq n$. Suppose $R^i \not\in R^{i+1}$. Then $R^i$ belongs to an $\in$-loop of circumference $n$, and there must be $x^{i-1}$ in $R^i$ in this loop. But $x^{i-1} \in R^i$ implies that $x^{i-1}$ cannot belong to any such loop. Thus we conclude $R^i \in R^{i+1}$. But $i$ was arbitrary. So there is an $\in$-loop of circumference $n$ consisting entirely of the $R^i$ and this clearly cannot happen. ■

It doesn't seem to be possible to spice up this proof to show (in first-order logic) that *none* of the $R^i$ exist. The nonexistence of $\{x : x \not\in^n x\}$ is a theorem of first-order logic that is stratifiable-mod-$n$, but I know of no globally stratifiable-mod-$n$ cut-free proof. This fact (if it is a fact) is almost certainly related to the fact (if it is a fact) that we cannot prove that $\{x : x \not\in^n x\}$ exists at no level (though we can show that it doesn't exist at all). If we had a proof of the nonexistence of $\{x : x \not\in^n x\}$ in FOL that was globally stratifiable-mod-$n$ then we could run it at any level and show that $\{x : x \not\in^n x\}$ exists at no level. The following reflection suggests that there is no such proof. Consider the two-lobed model with precisely one inhabitant in each lobe: a `yin` set that is a member of the `yang` set (but not the other way round). The `yang` set (but not the `yin` set) is a double-Russell class.

Consider the two-lobed structure depicted below: two objects in the left lobe $l_1$ and $l_2$, and two objects in the right lobe: $r_1$ and $r_2$. $l_1 \in r_1 \in l_1$; $l_2 \in r_2$, so that $l_2 = \emptyset$; $r_2 = \{l_2\}$; $r_1 = \{l_1\}$ and $l_1 = \{r_1\}$. It's a model of extensionality, and $r_2$ is the double Russell class on the right; there is no double Russell class on the left.




This does rather suggest that first-order logic holds no globally stratifiable-mod-2 nonexistence proof for the double Russell class... *and that this is true even if we allow cut.*

### 6.1 Rieger-Bernays Permutation methods for $\mathrm{TC}_n\mathrm{T}$

Rieger-Bernays methods generalise smoothly to $\mathrm{TC}_n\mathrm{T}$. R-B methods in NF enable one to obtain from any model of NF a new model which satisfies the same stratifiable sentences but tweaks the truth-values of some formulæ that are not stratifiable. In the $\mathrm{TC}_n\mathrm{T}$ context we have the same notion of stratifiable, but the role of non-stratifiable formulæ is played by formulæ that are stratifiable-mod-$n$. Formulæ that are frankly unstratified *don't enter into it* as the man in the parrot shop would say. The R-B method which we develop below for $\mathrm{TC}_n\mathrm{T}$ will enable us to obtain from a model $\mathfrak{M}$ of $\mathrm{TC}_n\mathrm{T}$ a model that satisfies the same stratifiable sentences as $\mathfrak{M}$ but satisfies *different* sentences that are merely stratifiable-mod-$n$.

It goes as follows. Let $\mathfrak{M}$ be a model of $\mathrm{TC}_n\mathrm{T}$. To each of the $n$ levels of $\mathfrak{M}$ associate an internal permutation $\tau$ of that level. Thus we have a *suite* of permutations. Then we declare a new membership relation between levels $i$ and $i+1$ by $x_i \in_{new} x_{i+1}$ iff $x_i \in \tau(x_{i+1})$. The relettering now proceeds as in the proof of Henson's lemma. For this we naturally need all the permutations in the suite to be setlike, just as in the original R-B setting. Realistically we can take them to be sets of the model.

Do we need a reference for henson's lemms?

Observe the two special cases: NF and TST. NF is the special case $\mathrm{TC}_1\mathrm{T}$. There is only permutation, and we are in the standard situation with Rieger-Bernays models for NF. TST is the case $\mathrm{TC}_\infty\mathrm{T}$ and we have infinitely many permutations, one for each level. In this case nothing happens, because there are no wellformed formulæ that this process could possibly change the truth-value of. No wonder nobody noticed it before!

If we want to preserve formulæ that are stratifiable-mod-$k$ then we require certain equations to hold between the permutations $\tau$ that we use. Consider $\mathrm{TC}_3\mathrm{T}$, the simplest case that is complicated enough to partake of the general flavour. We have a suite $\pi, \tau, \sigma$ of permutations. In the permutation model $x_1 \in x_2$ becomes $x_1 \in \tau(x_2)$; $x_2 \in x_3$ becomes $x_2 \in \pi(x_3)$ and $x_3 \in x_1$ becomes $x_3 \in \sigma(x_1)$. To reletter we have to rewrite $x_2 \in \pi(x_3)$ as $\tau(x_2) \in (j\tau) \cdot \pi(x_3)$

and then rewrite $x_3 \in \sigma(x_1)$ as $(j\tau) \cdot \pi(x_3) \in (j^2\tau) \cdot j\pi \cdot \sigma(x_1)$. If we want to be able to eliminate $\pi$, $\sigma$ and $\tau$ from formulæ that are stratifiable-mod-3 (but not stratifiable) then we will need $(j^2\tau) \cdot j\pi \cdot \sigma = \mathbb{1}$. Call this **The Equation For $n$.**

Very well: so we have a model of $\mathrm{TC}_n\mathrm{T}$, and we decorate it with permutations of each level. This R-B construction preserves all stratifiable expressions. What about expressions that are stratifiable-mod-$(n \cdot k)$? Then we have The Equation For $n \cdot k$. This is an equation $w = \mathbb{1}$ where $w$ is a product of $n \cdot k$ things, with $k$ occurrences of each permutation. Persisting for the moment with the $n = 3$ example, we find that if we want our suite of permutations to preserve formulæ that are stratifiable-mod-6, then we need $\tau$, $\sigma$ and $\pi$ to satisfy

$$(j^5\tau) \cdot (j^4\pi) \cdot (j^3\sigma) \cdot (j^2\tau) \cdot j\pi \cdot \sigma = \mathbb{1}.$$

This looks messy, but I think it is correct. needs editing

There is also the small matter of proving an analogue of the PHF theorem for $\mathrm{TC}_n\mathrm{T}$. We can obtain a model of $\mathrm{T\mathbb{Z}T}$ from a model of $\mathrm{TC}_n\mathrm{T}$ in the same was as we obtain one from a model of NF. The analogue will then say that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the same stratifiable sentences iff the two models of $\mathrm{T\mathbb{Z}T}$ obtained from them have stratimorphic ultrapowers. needs editing. PHF??

But what about stratifiable-mod-$n$? What condition on two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathrm{TC}_n\mathrm{T}$ corresponds to them satisfying the same formulæ that are stratifiable-mod-$n \cdot m$? I'm guessing it will be the following. Any model of $\mathrm{TC}_n\mathrm{T}$ can be turned into a model of $\mathrm{TC}_{m \cdot n}\mathrm{T}$ in an obvious way. So: let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two models of $\mathrm{TC}_n\mathrm{T}$. Obtain two models of $\mathrm{TC}_{m \cdot n}\mathrm{T}$. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the same formulæ that are stratifiable-mod-$n \cdot m$ iff these two models have isomorphic ultrapowers. I should be able to prove this but I am old and tired and I have multiple infarct dementia.

We should probably try to find something to say about expressions that, for every $n$, are logically equivalent to a formula that is stratifiable-mod-$n$. It's not true that all such sentences are stratified, because the axiom of counting is a counterexample. An interesting example of a sentence of this kind is "every wellfounded set is finite". It's not known if this allegation is invariant. However, at least some such expressions are not invariant. needs editing

### 6.2 Possible Equiconsistency of $\mathrm{TC}_n\mathrm{T}$ and NF

Two fundamental questions:

(i) Is $\mathrm{TC}_n\mathrm{T}$ equiconsistent with NF?

(ii) Are there models of $\mathrm{TC}_n\mathrm{T}$ that are not ambiguous? Equivalently, Are there models of $\mathrm{T\mathbb{Z}T}+\mathrm{Amb}^n$ that are not models of Amb?

I'm guessing that the answer to both is 'yes', but I have no idea how to prove it. Part of the trouble is that I can't think of a stratified formula for which Amb might fail while $\mathrm{Amb}^2$ holds. needs editing

Is $\mathrm{TC}_n\mathrm{T}$ equiconsistent with NF? One direction is easy. We can obtain a model of $\mathrm{TC}_n\mathrm{T}$ from a model of NF by making $n$ copies of the model of NF and decorating them appropriately. The obvious way is as follows. Let level $n$ of $\mathfrak{M}$ be $V \times \{n\}$ and let us declare that $\mathfrak{M} \models x_n \in y_{n+1}$ iff let $x_n = \langle x, n\rangle$ in let $y_{n+1} = \langle y, n+1\rangle$ in $x \in y$. $\mathfrak{M}$ is clearly an ambiguous model.

However one would not expect *every* model of $\mathrm{TC}_n\mathrm{T}$ to be ambiguous, because that would mean that $\mathrm{Amb}^n$ implies Amb, and that surely cannot be true. It would be nice to obtain a model of $\mathrm{TC}_2\mathrm{T}$ that violates ambiguity. A simple observation is that no model of $\mathrm{TC}_2\mathrm{T}$ can contain a Boffa atom in one lobe and a Boffa antiatom in the other. (A Boffa antiatom is a $b$ s.t. $(\forall x)(x \in b \longleftrightarrow b \notin x)$. No chance of such an object in a one-sorted universe, but here... perhaps) This means that if we can find a model $\mathfrak{M}$ that has in one lobe both a Boffa atom and a Boffa antiatom then $\mathfrak{M}$ must violate ambiguity, because an ambiguous model with a Boffa atom plus antiatom in one lobe must also contain a Boffa atom plus antiatom in the other, giving a Boffa atom in one lobe and a Boffa antiatom in the other and this is impossible, as we have just observed. needs editing. Boffa antiatoms are problematic

We can do this by a simple tweak of the obvious construction. Let $\tau$ be some permutation that adds a Boffa atom and a Boffa antiatom, such as $(\emptyset, B(\emptyset)) \cdot (V, \overline{BV})$. Then we set both `yin` and `yang` to be $V$ and we set $x_{\mathtt{yin}} \in y_{\mathtt{yang}}$ iff $x \in y$ and $x_{\mathtt{yang}} \in y_{\mathtt{yin}}$ iff $x \in \tau(y)$. Let us call this model $\mathfrak{M}$.

$\mathfrak{M}$ is clearly extensional. If we pinch ourselves to keep in mind that the comprehension axioms of $\mathrm{TC}_2\mathrm{T}$ are the fully stratifiable instances of comprehension and not the (larger) class of comprehension axioms that are stratifiable-mod-2 (Beware the double Russell class) then we can see that all the instances of comprehension for $\mathfrak{M}$ follow smoothly from comprehension in the model of NF in which we are working. (See also p 24.)

[One potentially useful piece of clarification.... What happens if we use $\tau$ on *both* lobes, so that we set $x_{\mathtt{yin}} \in y_{\mathtt{yang}}$ iff $x \in \tau(y)$ and $x_{\mathtt{yang}} \in y_{\mathtt{yin}}$ iff $x \in \tau(y)$? Clearly we do not get Boffa atoms plus antiatoms in *both* lobes – beco's we can't – but it might help to show what becomes of $V$ and $\emptyset$ in each lobe. They probably become something annoying that is *almost* an atom or an antiatom.]

However this construction does not resolve the question. Hitherto all discussions of ambiguity were in the context of TST. The scheme was: take any stratifiable formula $\phi \in \mathcal{L}(\in, =)$, decorate it with type indices on the variables, and assert biconditionals between the results. The point is that *all* formulæ of $\mathcal{L}(TST)$ arise from formulæ in $\mathcal{L}(\in, =)$ by this process of decoration. **However** $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$ has extra formulæ that can be decorated in this way, namely the formulæ that are stratifiable-mod-$n$. Ambiguity fails in the model $\mathfrak{M}$ that we have just considered, but the failure we have exhibited concerns not formulæ that arose from *stratifiable* formulæ of $\mathcal{L}(\in, =)$, but a formula that arose from a formula in $\mathcal{L}(\in, =)$ that was *stratifiable-mod-$n$*. I claim that $\mathfrak{M}$ satifies ambiguity for all formulæ that arose from *stratifiable* formulæ in $\mathcal{L}(\in, =)$. Let $\phi$ be any closed stratifiable formula of $\mathcal{L}(\in, =)$. Fix a stratification of it. This stratification awards every variable a decoration that is either an even natural or an odd natural. We can now turn $\phi$ into a formula of $\mathcal{L}(\mathfrak{M})$ in two ways: make every variable with an *even* decoration into a variable of type `yin` and every variable with an *odd* decoration into a variable of type `yang` or *vice versa*. But then in both these cases any variable $v$ that ever appears in a context "$\ldots \in \tau(v)$" only ever appears in such contexts, and so can be relettered.

If we think of the task of finding a model of $\mathrm{TC_2T}$ that is not ambiguous as the task of finding a model of TZT that satisfies $\mathrm{Amb}^2$ but does not satisfy Amb then it perhaps becomes clearer. This second task clearly remains undone.

It's an old result (it was in Forster's Ph.D. thesis, with a much improved proof by Crabbé [3] subsequently) that $\mathrm{T\mathbb{Z}T}$+ $\mathrm{Amb}^n$ refutes AC, and by essentially the same mechanism as does $\mathrm{T\mathbb{Z}T}$+ Amb. The best guess is that all the theories $\mathrm{TC}_n\mathrm{T}$ are equiconsistent with NF.

I noted above, in definition 5, that we have to make sure that our comprehension axioms are only those formulæ which become axioms of $\mathrm{T\mathbb{Z}T}$, lest we get Russell-style paradoxes. It might be worth thinking a bit about how one might cautiously relax this restriction to admit some more comprehension axioms. There is an analogue of *strongly cantorian* and although one obviously cannot allow the class of analogue-stcan sets to be a set (for the usual reasons concerning the Burali-Forti paradox) there doesn't seem to be any objection to the collection of *finite* analogue-stcan sets being a set.

Since "$x$ is wellfounded" can now be captured by a formula that is stratifiable-mod-$n$ and separation for wellfounded sets is safe for many expressions we should sort that out. In this next section we consider the property "$\iota^n \restriction x$ exists" which is stratifiable-mod-$n$.

# 7 Modulo-$n$ Analogues of *strongly cantorian*

## 7.1 Analogues in NF

In this section we work in NF.

"$\iota^n \restriction x$ exists" is an analogue of $x$ *is strongly cantorian*. Lots of things to be said about it. Is this generalisation of strong cantorian-ness a good notion of small set? In the categorial sense, that is?

I noticed years ago the fact that although the existence of $\iota \restriction x$ clearly implies the existence of $\iota^n \restriction x$, the converse does not seem to hold. If $\iota^2 \restriction x$ exists then certainly $x \sqcup \iota``x$ is cantorian but that (and its analogues for $n > 2$) seem to be as far as one can go. It would appear that, in principle, there might be sets $x$ s.t. $\iota^n \restriction x$ exists for some $n$ but which are nevertheless not strongly cantorian.

The property "$\iota^n \restriction x$ exists" is inherited by subsets in the same way that strong-cantorianness is, so it is an *analogue* of 'strong cantorian' rather than a mere *weakening* of it – unlike 'cantorian' which (being a mere weakening) is not inherited by subsets in the same way.

The possible existence of such sets is worth noting in the present context, since for them one can prove an analogue of *subversion of stratification* for formulæ that are stratifiable-mod-$n$.

Subversion of stratification says that, if $M$ is a strongly cantorian set, and $\phi$ an arbitrary formula, then $\{x \in M : \phi^M(x)\}$ exists. ($\phi^M$ is the result of restricting all quantifiers in $\phi$ to $M$.) The analogue here would say that, if $\iota^n \restriction M$ exists and $\phi$ is stratifiable-mod-$n$, then $\{x \in M : \phi^M(x)\}$ exists. Of course this will hold in $\mathrm{TC}_n\mathrm{T}$ ... which may be the correct setting for this observation: $\mathrm{TC}_n\mathrm{T}$ has subversion of stratification for $x$ s.t. $\iota^n \restriction x$ exists, in the sense that the following holds.

**Remark 6**
*If $\iota^n \restriction M$ exists, and $\phi$ is stratifiable-mod-$n$ then $\{y \in M : \phi^M(y)\}$ exists.*

*Proof:* ■

Should really write out a proof.

Just as subversion for strongly cantorian sets gives us interpretations into (extensions of) NF of fully unstratified set theories, subversion for sets $x$ for which $\iota^n \restriction x$ exists will give us interpretations into (extensions of) NF of set theories satisfying syntactic contraints correspondingly less onerous than full stratification.

Subversion of stratification enables us to cutely finitise the restriction of the scheme of $\Delta_0$ separation to formulæ that are stratifiable-mod-$n$. We know how to finitely axiomatise stratifiable $\Delta_0$ separation (see the second edition of the monograph [8]), and we can get full $\Delta_0$ separation from that axiomatisation simply by adding the existence of $\iota \restriction x$ for all $x$. The obvious thing to do is augment the kit of rudimentary functions by adding a new rudimentary function which gives $\iota^n \restriction x$, and then rely on subversion.

Does this open up a vein of novel, more delicate, relative consistency proofs? Possibly, but not if we are adopting an axiom of infinity: the assumption that there is an (infinite) $x$ s.t. $\iota^n \restriction x$ exists is as strong as the assumption that there is an infinite strongly cantorian set. This triviality is worth minuting because we will make use of it elsewhere (see p. 6).

**Remark 7**
*(i) If $x$ is wellorderable and $\iota^n \restriction x$ exists then $x$ is strongly cantorian.*
*(ii) If there is an infinite $x$ and a concrete $n$ such that $\iota^n \restriction x$ exists then the axiom of counting holds.*

*Proof:*
(i) If $x$ is a wellorderable set s.t. $\iota^n \restriction x$ exists then the order type of any worder of $x$ is certainly going to be less than all of $\Omega$, $\Omega_1$ ... [6], so we can assume without loss of generality that $x$ is an initial segment $X$ of the ordinals. This

[6] $\Omega$ is the order type of the set of ordinals; $\Omega_1 = T\Omega$, and so on.

means that $\iota^n \restriction X$ exists, and that in turn means that $T^n \restriction X$ exists, and that in turn means that we can prove by induction on the ordinals that $T^n \restriction X$ is the identity. So, for every $\alpha \in X$, $T^n\alpha = \alpha$. For every ordinal $\alpha$ (and so in particular for every $\alpha \in X$) we have $\alpha = T\alpha \vee \alpha < T\alpha \vee \alpha > T\alpha$. The third disjunct implies (apply $T$ to both sides) $T^2\alpha < \alpha$ giving $\alpha < T\alpha < \ldots T^n\alpha$ contradicting $T^n\alpha = \alpha$; the second disjunct is refuted similarly. So $T \restriction X$ exists beco's it is the identity, so $\iota \restriction X$ exists as well.

(ii) This property "$\iota^n \restriction x$ exists" is preserved by power set as well as by subset, so if there is even one infinite set which has it then $\mathbb{N}$ will have it as well. (Just as: $\mathbb{N}$ is strongly cantorian if there is even one infinite strongly cantorian set). But $\mathbb{N}$ is wellordered, so we can apply part (i). ■

The other direction (inferring "$\iota^n \restriction \mathbb{N}$ exists" for arbitrary concrete $n$ from the axiom of counting) is easy. Thus, for every (concrete) $n$, the axiom of counting is equivalent modulo NF to a formula that is stratifiable-mod-$n$. If $\phi$ is, for each $n$, equivalent (modulo NF) to something that is stratifiable-mod-$n$ must it be (NF)-invariant? No, because the property of being a wellfounded set is, for each $n$, equivalent (modulo NF) to something that is stratifiable-mod-$n$ but is not invariant. But what about closed formulæ? No, that doesn't work either, as we shall see below.

Let $\mathrm{Mac}_n$ be Mac with separation restricted to formulæ that are $\Delta_0$ and stratifiable-mod-$n$. Analogues of the result in [12] to the effect that Mac + TCl can be interpreted into KF can be obtained, saying that $\mathrm{Mac}_n$ + TCl can be interpreted into KF, but these results are weaker than the result in [12]. However these refined constructions could turn out to be useful should there turn out to be interesting theories of the form $\mathrm{Mac}_n \cup \{A\}$ (where $A$ is some formula not a theorem of Mac). However no such examples leap to mind. Not to the authors' mind anyway: $\exists$NO might have sounded like a starter but is is inconsistent with the existence of $\iota^n \restriction x$ for all $x$. (This last follows from remark 7 part (i).)

The upshot of this is that $\exists$NO is incompatible with $\mathrm{Mac}_n$, the point being that $\iota^n \restriction$(the representative set of wellorderings) would exist and that the quotient would be strongly cantorian.

**Lemma 1**
*For all concrete $n$ and $k$, $(\forall x)(\iota^n \restriction x$ exists$)$ implies $(\forall x)(\iota^{n \cdot k} \restriction x$ exists$)$.*

*Proof:*
We know that $\mathtt{RUSC}(R)$ always exists, so $\mathtt{RUSC}^k(R)$ exists for all $R$ and all concrete $k$, so $\mathtt{RUSC}^k(\iota^n \restriction x)$ exists and so $\iota^n \restriction x$ composed with $\mathtt{RUSC}^n(\iota^n \restriction x)$ exists, and that is $\iota^{n \cdot 2} \restriction x$. And so on for all the other multiples of $n$. ■

## 7.2 Ambiguity in $\mathrm{TC}_n\mathrm{T}$

Take a simple example: $\mathrm{TC}_2\mathrm{T}$. Since every formula that is stratifiable-mod-4 is also stratifiable-mod-2 we can assert in the language of $\mathrm{TC}_2\mathrm{T}$ that the `yin` collection that would be the quartic Russell class $\{x : x \not\in^4 x\}$ exists. Ambiguity for formulæ that are stratifiable-mod-4 would then say that the corresponding `yang` set exists. See the discussion on page 17.

It might be an idea to write out a proof that the quartic Russell classes $\{x : x \not\in^4 x\}$ cannot all exist. Put up or shut up

If we are right about all ambiguity$^n$ schemes being of equal consistency strength then it should be easy to prove the consistency of $\mathrm{TC}_n\mathrm{T}$ + Ambiguity for formulæ that are stratifiable-mod-$(m \cdot n)$ relative to $\mathrm{TC}_n\mathrm{T}$. Yeah right. needs editing

## 7.3 CO models for $\mathrm{TC}_n\mathrm{T}$

It is simplicity itself to cook up a CO model of (the version of) $\mathrm{TC}_n\mathrm{T}$ that corresponds to AST. (For definition of AST and more on CO models in general see [10].) Let $\langle \mathbb{N}, E\rangle$ be the standard Oswald model. Define a new relation $E'$ on $\mathbb{N}$ by

$$2n\, E'\, (2m+1) \text{ iff } n\, E\, m$$

and

$$(2n+1)\, E'\, 2m \text{ iff } n\, E\, m.$$

That way even numbers are `yin` and odd numbers are `yang`. I think the double Russell class will turn out to contain precisely the wellfounded sets... but this will need to be checked. It's clear how to do the same for $\mathrm{TC}_n\mathrm{T}$ for $n > 2$. You partition $\mathbb{N}$ into the $n$ residue classes mod $n$ and you say that $i$ is a member of $j$ in the new sense if $i + 1 \equiv j \bmod n$ and $(i \texttt{ DIV } n)\ E\ (j \texttt{ DIV } n)$. needs editing

Of course there is nothing special about $E$. We can do this for any Oswald model at all. What we might be able to do is get a model of the AST version of $\mathrm{TC}_2\mathrm{T}$ with a Boffa antiatom in one lobe but not in the other. It might be an instructive exercise to write this out in some detail. Check this: aren't Boffa antiatoms problematic?

We'll have two copies of $\mathbb{N}$: `yin` naturals and `yang` naturals. And we'll put a Boffa antiatom into level `yang` but not into level `yin`. In $n$ is a `yin` natural and $m$ a `yang` natural then we ordain than $m$ is a member of $n$ in the new sense iff $m\, E\, n$, where $E$ is the membership relation of the Oswald model. Membership of `yang` naturals echoes the construction of CO models containing moieties. You look at `yang` naturals mod 4: that is to say, peel off the two least significant bits of a `yang` natural $m$ and use them as a flag, which of course is $0, 1, 2$ or $3$.

> If the flag is 0 then we say $n$ belongs to $m$ in the new sense iff the $n$th bit of the truncation is 1;
>
> If the flag is 1 then we say $n$ belongs to $m$ in the new sense iff the $n$th bit of the truncation is 1;
>
> If the flag is 2 then we say $n$ belongs to $m$ in the new sense iff (the $n$th bit of the truncation is 1 iff $n$ belongs to the complement of the Boffa antiatom);
>
> If the flag is 3 then we say $n$ belongs to $m$ in the new sense iff (the $n$th bit of the truncation is 1 iff $n$ belongs to the Boffa antiatom).

But questions of whether or not any given `yin` $n$ belongs to the `yang` Boffa antiatom are answered by examining whether the Boffa antiatom is a member of $n$. And membership of `yang` sets in `yin` sets is unproblematic.

### 7.4 Generalise a Result of Specker?

Specker shows that in the situation where our language admits an automorphism $*$ of order 2, a conjunction of finitely many assertions of the form $\phi \longleftrightarrow \phi^*$ is another expression of that form. See Chad Brown's discussion of this question. Can we do anything similar here? Does it matter?

needs editing

Perhaps chop this doc into two at this point

## 8 Applications to Duality

Spend some time investigating the role of weak versions of AC. $\text{AC}_2$ presumably doesn't follow but it's worth a try.

> Fit this in somewhere
> Must show that if there is a polarity it's universal for involutions without fixed points. Well, actually no, co's that would imply that all polarities are conjugate, which sounds a bit restrictive. What happens if there is more than one polarity? If they are both universal then they're conjugate.
>
> Must show that if $AC_2$ *fails* and $\tau$ is universal for involutions without fixed points then so is $j\tau \cdot c$. First step is to show that in these circumstances $j\tau \cdot c$ has no transversal. The second step is to show that it is universal.
>
> ¿How easy is it going to be to show that $\tau \leq j\tau \cdot c$?
>
> Remember that there are only $T|V|$ pairs, so no involution can have more than $T|V|$ pairs. Nothing to say that it can't have fewer. But if it has fewer it has no transversal!
>
> If it's the wrong size it lacks a transversal. Is the converse true? Probably not. Can we show that if there is a partition into pairs that lacks a transversal then there is a right-sized one that lacks a transversal? Well we can add or delete ordered pairs to obtain a right-sized set of pairs without a choice function. Will that do?
>
> What sort of involutions-without-fixedpoints lack transversals?
>
> The set of involutions-without-fixedpoints-or-transversals are an upward-closed subset.
>
> Hmm. Suppose there is a partition-into-pairs that is smaller than $T|V|$. It lacks a transversal. But then anything Bowler-above it also lacks a transversal. There is also a universal partition-into-pairs. So it lacks transversals if anything does.
>
> antimorphisms form a torsor

The special case of stratification-mod-$n$ which will concern us here is $n = 2$. The context throughout this section is NF.

**Definition 6** *The dual $\phi^{\circlearrowleft}$ of a formula $\phi$ is the formula obtained from $\phi$ by replacing all occurrences of '$\in$' in $\phi$ by '$\notin$'.*

It has been known for some time that $\phi \longleftrightarrow \phi^{\circlearrowleft}$ is a theorem of NF whenever $\phi$ is a closed stratifiable formula. Permutation models can be found in which $\phi \longleftrightarrow \phi^{\circlearrowleft}$ *fails* for some unstratifiable $\phi$, but it remains an open question whether or not there are models in which $\phi \longleftrightarrow \phi^{\circlearrowleft}$ *holds* for all $\phi$. The natural conjecture is that there should be such models.

We do not prove the full conjecture here but we can prove the relative consistency of the scheme $\phi \longleftrightarrow \phi^{\circlearrowleft}$ at least for all $\phi$ that are stratifiable-mod-2. This will be theorem 6 below, and it is the principal aim of this section to prove it.

However, in preparation for theorem 6 we need to do a lot of bush-clearing in regard to NF's theory of permutations (and specifically involutions) of $V$, and this necessitates a few subsections of prolegomena.

## 8.1 Transversals, Partitions, Conjugacy and the Axiom of Choice for sets of pairs

First we reflect that the duality scheme might in principle be witnessed by the existence of an antimorphism.

**Definition 7**

- *A* **transversal** *for a disjoint family is a set that meets every member of the family on a singleton.*
- *An* **antimorphism** *is a permutation $\tau$ of $V$ satisfying*

$$(\forall x, y)(x \in y \longleftrightarrow \tau(x) \notin \tau(y)).$$

- *An antimorphism that is an involution is a* **polarity**.

Clearly if there is an antimorphism then duality follows.

Clearly in principle there is the possibility of antimorphisms of infinite order or any finite *even* order. However for the moment we will consider the possibility only of polarities. This restriction would appear to be of little cost, since it seems highly implausible that there might be antimorphisms but no polarities, and polarities are so much easier to think about.

Bowler [2] has found an injection $i$ from the set of pairs into the set of singletons: send $\{x, y\}$ to $\{(x \times y) \cup (y \times x)\}$. This enables us to infer (2) from (1):

1. Every set of disjoint pairs has a choice function;
2. Every set of pairs has a choice function.

Let $P$ be a set of pairs. We desire a choice function for it, but we know only (1) – not (2). The set

$$\{p \times i(p) : p \in P\}$$

is a family of disjoint pairs and therefore, by (1), has a choice function, $f$. We can recover a choice function $f^*$ for $P$ by $f^*(p) =: \texttt{fst}(f(p \times i(p)))$. ■

We will also need the equivalence of (3) and (4):

3. Every partition of $V$ into pairs has a choice function;

4. Every set of disjoint pairs has a choice function.

If we are given a set of pairs we can make disjoint copies of it by the trick we used earlier. In fact – by using an $i$ whose range is a moiety[7] of singletons – we can ensure that the sumset $\bigcup P$ of the disjoint family $P$ of pairs we construct by this method has a complement that is the same size as $V$. The complement $V \setminus \bigcup P$ therefore has a partition $\mathbb{P}'$ into pairs. Then $P \cup \mathbb{P}'$ is a partition of $V$ into pairs. Any transversal for this partition will give us a choice function for the partition we started with.

Two more propositions:

5. Whenever we partition $V$ into pairs we get the same number of pairs;

6. Whenever we partition $V$ into pairs the two partitions are conjugate.

It turns out that (6) is equivalent to $AC_2$. (I mention 5 only as a foil, lest a reader think i'm talking about 5 when I am in fact talking about 6).

- $AC_2 \to 6$

Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two partitions of $V$ into pairs. By $AC_2$ we have a transversal $S$ for $\mathbb{P}_1$, and $\mathbb{P}_1$ is obviously a bijection between $S$ and $V \setminus S$. So $|S| = |V|$ and $|\mathbb{P}_1| = T|V|$. Naturally we argue for $\mathbb{P}_2$ in the same way. So there is a bijection $\pi$ between $\mathbb{P}_1$ and $\mathbb{P}_2$. For each $p \in \mathbb{P}_1$ there are precisely two bijections between $p$ and $\pi(p)$ and we use $AC_2$ to pick one. The union of all such chosen bijections is a permutation conjugating $\mathbb{P}_1$ and $\mathbb{P}_2$.

- $6 \to AC_2$.

Assume 6. If $\mathbb{P}$ is a partition of $V$ into pairs then by 6 it will be conjugate to the partition $\{\{x, V \setminus x\} : x \in V\}$. That is to say, there is a permutation $\pi$ of $V$ such that, for all $p \in \mathbb{P}$, $\pi“p$ is a pair $\{x, V \setminus x\}$. But clearly the partition $\{\{x, V \setminus x\} : x \in V\}$ has a choice function $f$ (“pick the element that contains $\emptyset$”) so the choice function for $\mathbb{P}$ that we want is $p \mapsto \pi^{-1}(f(\pi“p))$. ■

So we have established:

**Remark 8** *The following are equivalent:*

*Every set of pairs has a choice function ($AC_2$);*
*Every set of disjoint pairs has a choice function;*
*Any two partitions of $V$ into pairs are conjugate;*
*Every partition of $V$ into pairs has a choice function.*
*Any two equinumerous partitions of $V$ into pairs are conjugate*

Here's a proof that the last bit implies $\mathrm{AC}_2$.

Let $\pi$ be a set of pairs without a choice function. Without loss of generality the pairs are disjoint. Take the disjoint union of $\bigcup \pi$ with $V$. The result is the same size as $V$ and can be canonically split into pairs using $c$ (on the copy of $V$) and $\pi$ (on the copy of $\bigcup \pi$). Copy this over into a partition of $V$ into

[7] Moieties are supposed to be the same size as $V$

pairs. We have $T|V|$-many pairs, which is the same as the number of pairs in the partition corresponding to $c$. So – *if any two partitions of $V$ into the same number of pairs are conjugate* – then this $\pi$ must have a choice function. But $\pi$ was arbitrary. ∎

If we partition $V$ into pairs how many do we get? No more than $T|V|$ (by this result of Bowler's in [2]) but can we get fewer? We must try to connect this with the question of whether or not $|V|$ is decomposable.

**Definition 8**
*An involution with no fixed points and no transversal set is* **bad**.

The thinking behind this censorious terminology is that an involution without fixed points is a partition of $V$ into pairs and will have a transversal as long as $AC_2$ holds. Thus we stigmatise involutions with no fixed points and no transversal as *bad* beco's their existence contradicts $AC_2$.

Involutions are peculiar among permutations of $V$ in that they can be thought of straightforwardly as partitions whose pieces are of size at most two. We will make much use of this freedom.

I assume the reader can work out for themselves that every polarity is a bad involution.

Can we say anything about the cardinality of a polarity tho'rt of as a set of pairs?

**Lemma 2** *Any two involutions-without-fixed-points whose corresponding partitions-of-$V$-into-pairs have transversals are conjugate.*

*Proof:*
First we establish that if $\mathcal{T}$ is a transversal for a partition $\mathbb{P}$ of $V$ into pairs then its cardinality is $|V|$. Clearly $|\mathbb{P}| = T|\mathcal{T}|$, since we can send each piece of $\mathbb{P}$ to the unique singleton $\subset \mathcal{T}$ that meets it. Observe that there is a bijection between $\iota``V$ and $\mathbb{P} \times \{0, 1\}$, as follows. For each $x$ there is a unique $p_x \in \mathbb{P}$ with $x \in p_x$. If $x \in \mathcal{T}$ we send $\{x\}$ to $\langle p_x, 0\rangle$; if $x \notin \mathcal{T}$ we send $\{x\}$ to $\langle p_x, 1\rangle$.

Finally if $\pi_1$ and $\pi_2$ are two involutions-without-fixed-points equipped with transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, then not only do we have $|\mathcal{T}_1| = |\mathcal{T}_2| = |V|$ but $\pi_1$ and $\pi_2$ are conjugate, as follows. $\mathcal{T}_1$ and $\mathcal{T}_2$ are in bijection, by a map $\pi^*$, say. Any such $\pi^*$ can be extended to a permutation $\pi$ of the universe by adding all the ordered pairs $\langle \pi_1(x), \pi_2(\pi^*(x))\rangle$ for $x \in \mathcal{T}_1$.

∎

Some minor points:

(i) The proof of lemma 2 as given above tells us nothing about permutations that conjugate $\pi_1$ and $\pi_2$ beyond the fact that they exist. However the construction is effective and can be mined for more information. In lemmas 3 and 8 we consider a particular case in which we need more information and we go into more detail.

(ii) Notice that in lemma 2 the assumption on the two involutions is that the corresponding partitions have transversals. It is not the weaker assumption that the corresponding partitions are the same size.

*Might* it be possible to prove in NF that any two partitions of $V$ into pairs are the same size. . . ? After all – as mentioned above – Nathan Bowler [2] has shown us a proof in NF that there are as many pairs as singletons.

Sadly no, not unless NF $\vdash$ $AC_2$.

What is the status of "All partitions of $V$ into pairs are equinumerous"?

We now need Nathan Bowler's fruitful idea of a *universal involution*. That in turn relies on a notion of permutation morphism due to Bowler:

**Definition 9**

- *For permutations $\sigma$ and $\tau$ of sets $X$ and $Y$, a* **map of permutations** *from $\sigma$ to $\tau$ is a function $\pi : X \to Y$ such that $\pi \cdot \sigma = \tau \cdot \pi$.*
- *If $\pi$ is injective, we call it an* **embedding of permutations***.*
- *An involution is* **universal** *if every involution embeds into it.*
- *We will write "$\sigma \leq_B \tau$" to say that there is an embedding of permutations from $\sigma$ to $\tau$.*

In all the cases of interest to us we have $X = Y = V$ and $\pi$ will be injective, but we shouldn't forget that Bowler's definition is more general.

Think of a permutation as a digraph wherein every vertex has indegree one and outdegree one, and loops at vertices are allowed. An embedding-of-permutations must send $n$-cycles onto $n$-cycles, and when you look at it like that the Cantor-Bernstein–style theorem below (lemma 3) becomes much more obvious.

For the moment we need definition 9 only for involutions, and we will speak of *involution-embeddings* or *embeddings of involutions*. In due course we will prove (lemma 4) that there are universal involutions, and give examples.

We will need the following analogue of Cantor-Bernstein for embeddings-of-permutations.

**Lemma 3** *BCB (Bowler's Cantor-Bernstein lemma)*

*If $\sigma$ is a permutation of $X$ and $\tau$ a permutation of $Y$ with $\sigma \leq \tau \leq \sigma$ then $\sigma$ and $\tau$ are conjugate.*

*Proof:* (Bowler, edited by tf)

Suppose $\sigma \leq \tau$ in virtue of $\rho : X \hookrightarrow Y$ and $\tau \leq \sigma$ in virtue of $\pi : Y \hookrightarrow X$. Consider the map $\mathcal{P}(X) \hookrightarrow \mathcal{P}(X)$ defined by $S \mapsto X \setminus \rho``(Y \setminus \pi``S)$.
By Tarski-Knaster this map has a least fixed point, which we will call $P$.
Then the map $X \hookrightarrow Y$ given by $\pi \restriction P \cup \rho^{-1} \restriction X \setminus P$ conjugates $\sigma$ to $\tau$. ■

Notice that the map that conjugates $\sigma$ and $\tau$ has a stratifiable definition in terms of them, so if they are definable it is too, and so is its least fixed point. It won't matter that there is a least fixed point, but it will matter that there is a fixed point that is definable in terms of $\rho$ and $\pi$, and the lfp is one such.

In fact – for the moment – we will need lemma 3 only for involutions.

**Corollary 1** *Any two universal involutions of $V$ are conjugate.*

■

Lemma 3 is telling us that the intersection of the quasiorder $\leq_B$ and its converse $\geq_B$ is the equivalence relation of conjugacy. This makes it sort-of OK to abuse notation by additionally using '$\leq_B$' to denote the partial ordering induced on the quotient. The quotient is a directed poset because of disjoint unions of copies of $V$. Is it an upper semilattice? It certainly supports a $+$ operation, but whether or not $[\sigma] + [\tau]$ is the sup of $[\sigma]$ and $[\tau]$ is another matter!

The following elementary facts will loom large.

**Remark 9**
*(i) Conjugacy is a congruence relation for $j$;*
*(ii) Conjugacy is a congruence relation for $\tau \mapsto j\tau \cdot c$;*
*(iii) $j$ preserves $\leq_B$.*

*Proof:*

(i) is obvious (and skew-conjugacy, too, is a congruence relation for $j$, though that is not as important here).

(ii) We will prove that $\sigma \leq \tau \to j\sigma \cdot c \leq j\tau \cdot c$ Assume $\sigma \leq \tau$ so that there is $\pi$ s.t.

$$\pi\sigma = \tau\pi$$

so

$$j\pi j\sigma = j\tau j\pi$$

and

$$j\pi j\sigma \cdot c = j\tau j\pi \cdot c$$

$$j\pi j\sigma \cdot c = j\tau \cdot c \cdot j\pi$$

which implies

$$j\sigma \cdot c \leq j\tau \cdot c$$

So $\sigma \mapsto j\sigma \cdot c$ respects conjugacy as desired. Is it injective on conjugacy classes? (ii) will be useful when we come to consider antimorphisms

For (iii) Observe that if $\pi$ is an embedding of permutations from $\sigma$ to $\tau$ then $j(\pi)$ is an embedding of permutations from $j(\sigma)$ to j($\tau$).

■

We will need this in the proof of the second part of lemma 4.

We begin by giving some examples of universal involutions of $V$.

**Lemma 4** *(Bowler, [1])*
*For all $i > 0$, $j^i(c)$ is a universal involution.*

*Proof:*

First we prove that $j(c)$ is universal.

There are bijections $V \longleftrightarrow \{x : \emptyset \not\in x\}$; in what follows fix $\theta$ to be one of them – it won't matter which.

For any involution $\sigma$ of any set $X$ we define an embedding of involutions $\pi$ from $\sigma$ to $j(c)$ by

$$\pi : x \mapsto j(\theta)(x) \cup j(c \cdot \theta)(\sigma(x)).$$

Observe that the two sets on the RHS are disjoint, since – by choice of $\{x : \emptyset \not\in x\}$ – the ranges of $\theta$ and $c \cdot \theta$ are disjoint.

The function $\pi$ is injective, with left inverse $y \mapsto j(\theta^{-1})(\{z \in y : \emptyset \not\in z\})$.

To see that $\pi$ is a map of involutions from $\sigma$ to $j(c)$ we calculate as follows:

| | | |
|---|---|---|
| (1) | $(j(c) \cdot \pi)(x) = j(c)(\pi(x))$ | Expand $\pi(x)$ to get |
| (2) | $= j(c)[j(\theta)(x) \cup j(c \cdot \theta)(\sigma(x))]$ | distribute $jc$ over $\cup$ to get; |
| (3) | $= j(c)(j(\theta)(x)) \cup jc \cdot j(c \cdot \theta)(\sigma(x))$ | |
| (4) | $= j(c \cdot \theta)(x) \cup j(c \cdot c \cdot \theta)(\sigma(x))$ | |
| (5) | $= j(c \cdot \theta)(x) \cup j(\theta)(\sigma(x))$ | reorder the set unions to get |
| (6) | $= (j\theta)(\sigma(x)) \cup j(c \cdot \theta)(x)$ | which gives |
| (7) | $= (j\theta)(\sigma(x)) \cup j(c \cdot \theta)(\sigma(\sigma(x)))$ | beco's $\sigma$ is an involution; |
| (8) | $= \pi(\sigma(x))$ | |

Two observations:

For the main result we argue as follows.

Clearly any involution into which a universal involution can be embedded is also universal, and any involution conjugate to a universal involution is again universal.

Since $j(c)$ is universal, there is an embedding of $c$ into $j(c)$. This lifts to embeddings of $j^i(c)$ into $j^{i+1}(c)$, and composing these embeddings we get embeddings of $j(c)$ into $j^i(c)$ for any $i \geq 1$. Thus $j^i(c)$ is universal for any $i \geq 1$.

■

> It might be an idea to properly write out a proof that $j$ lifts in this way. We should also check that $j(c)$ and $j^2(c)$ are conjugate and do it *by hand* as it were, so that we can tell whether or not they are conjugated by anything definable.

So $j(c)$ is a universal involution.

Notice that this means that $jc$ and $j^2c$ are conjugate and therefore give rise to a permutation model containing an automorphism that is an involution, as follows.

| | |
|---|---|
| $c \leq jc$ beco's $jc$ is universal. | Lift by $j$ |
| $jc \leq j^2c$ | but in any case we have |
| $j^2c \leq jc$ | beco's $jc$ is universal. But then |
| $j^2c$ and $jc$ are conjugate | by BCB – lemma 3. |

For this to work we needed not only that there should be a universal permutation of that flavour, but also that that universal permutation should be $j$ of something.

I think i now see how to prove the existence of permutations that are universal for degrees other than 2. I think this is the way to do it

The key is to understand that what Nathan is doing in definition 9 is *pipelining*. Here is how to have a permutation that is universal for permutations of degree $n$.

We split $V$ into $n$ moieties $M_0 \cdots M_{n-1}$ and we have a permutation $\tau$ with $\tau^n = \mathbb{1}$ s.t. for each $0 \leq i < n-1$, $\tau``M_i = M_{i+1}$, so that $\tau``M_{n-1} = M_0$. We also have $\theta : V \longleftrightarrow M_0$.

The idea is that $j\tau$ will be universal for permutations of degree $n$.

Let $\sigma$ be a permutation with $\sigma^n = \mathbb{1}$. We then define

$$\pi(x) = (j\theta)(x) \cup (j\tau)(j\theta)\sigma(x) \cup (j\tau)^2(j\theta)\sigma^2(x) \cdots (j\tau)^{n-1}(j\theta)\sigma^{n-1}(x)$$

So $(j\tau)\pi(x)$ will be

$$(j\tau)[(j\theta)(x) \cup (j\tau)(j\theta)\sigma(x) \cup (j\tau)^2(j\theta)\sigma^2(x) \cdots (j\tau)^{n-1}(j\theta)\sigma^{n-1}(x)]$$

$$= (j\tau)(j\theta)(x) \cup (j\tau)(j\tau)(j\theta)\sigma(x) \cup (j\tau)(j\tau)^2(j\theta)\sigma^2(x) \cdots (j\tau)(j\tau)^{n-1}(j\theta)\sigma^{n-1}(x)$$

$$= (j\tau)(j\theta)(x) \cup (j\tau)^2(j\theta)\sigma(x) \cup (j\tau)^3(j\theta)\sigma^2(x) \cdots \underline{(j\tau)^n}(j\theta)\sigma^{n-1}(x)$$

The underlined expression simplifies to $\mathbb{1}$.

$$= (j\tau)(j\theta)(x) \cup (j\tau)^2(j\theta)\sigma(x) \cup (j\tau)^3(j\theta)\sigma^2(x) \cdots \underline{(j\theta)\sigma^{n-1}(x)}$$

move the underlined bit to the front.

$$= (j\theta)\sigma^{n-1}(x) \cup (j\tau)(j\theta)(x) \cup (j\tau)^2(j\theta)\sigma(x) \cup (j\tau)^3(j\theta)\sigma^2(x) \cdots$$

which is $\pi \cdot \sigma^{n-1}(x)$.

So $\pi$ is an embedding-of-permutations from $\sigma^{n-1}$ to $j\tau$. But $\sigma^n = \mathbb{1}$ by hypothesis, so $\pi$ is an embedding-of-permutations from $\sigma^{-1}$ to $j\tau$. So we should replace '$\sigma$' with '$\sigma^{-1}$' in the definition of $\pi$.

Observe that this universal permutation – $j\tau$ – is $j$ of something and will therefore give rise to an automorphism in a permutation model.

So, for each $n \in \mathbb{N}$, there is a permutation which is universal for permutations of degree $n$. So there is a disjoint union of them, and it will be universal for the permutations with no infinite cycles. That doesn't seem to be enough to give us a universal permutation.

But perhaps we can get a permutation that is universal for permutations with no finite cycles by refining the pipelining.

So here goes, reusing Nathan's Greek letters...

Find a partition of $V$ into moieties $\{M_i : i \in \mathbb{Z}\}$ and a permutation $\tau$ of $V$ that maps each $M_i$ onto $M_{i+1}$. And $\theta$ is a map $V \longleftrightarrow M_0$. $j\tau$ is going to be a universal permutation. Then, given $\sigma$, declare

$$\pi(x) = \bigcup_{n \in \mathbb{Z}} (j\tau^n)(j\theta)\sigma^n(x)$$

whence

$$(j\tau)\pi(x) = \bigcup_{n \in \mathbb{Z}} (j\tau^{n+1})(j\theta)\sigma^{n+1}(\sigma^{-1}x)$$

renumbering

$$(j\tau)\pi(x) = \bigcup_{n+1 \in \mathbb{Z}} (j\tau^{n+1})(j\theta)\sigma^{n+1}(\sigma^{-1}x)$$

renumbering

$$(j\tau)\pi(x) = \bigcup_{n \in \mathbb{Z}} (j\tau^n)(j\theta)\sigma^n(\sigma^{-1}x)$$

which is $\pi(\sigma^{-1}(x))$, so $\pi \cdot \sigma^{-1}(x) = j\tau \cdot \pi(x)$ which says that $\pi$ witnesses $\sigma^{-1}$ embedding into $j\tau$.

(So we should really replace $\sigma$ by $\sigma^{-1}$ in the definition of $\pi$.)

We didn't put any conditions on $\pi$ he upshot is that this $j\tau$ seems to be universal for *all* permutations. How can this be? Surely the embedding must send $n$ cycles to $n$-cycles?! What goes wrong with the proof if $\sigma$ is of finite order? I think the answer is that there is no problem beco's $j\tau$ has cycles of all finite sizes. We have stumbled upon a genuinely universal permutation:

| | |
|---|---|
| $\tau \leq j\tau$ beco's $j\tau$ is universal. | Lift by $j$ |
| $j\tau \leq j^2\tau$ | but in any case we have |
| $j^2\tau \leq j\tau$ | beco's $j\tau$ is universal. But then |
| $j^2\tau$ and $j\tau$ are conjugate | by BCB – lemma 3. |

If there is a permutation that is universal for permutations with no infinite cycles we might expect it to give rise to an automorphism with arbitrarily large finite cycles but no infinite cycles. This would contradict countable choice. So presumably the universal permutation whose construction we have outlined above is not $j$ of anything.

Suppose there were a permutation that is universal for permutations-with-no-infinite-cycles that was $j\tau$ for some $\tau$. We would like to get a contradiction with countable choice. Then $\tau$ has no infinite cycles, whence $\tau \leq j\tau$. This gives $j\tau \leq j^2\tau$. But there we run into the sand. We want $j^2\tau \leq j\tau$ but there seems no way of proving it.

**REMARK 10** *There is a universal involution-without-fixpoints.*

*Proof:*

Of course (as we have seen) if $AC_2$ holds then there is a *unique* conjugacy class of involutions-without-fixpoints: *all* involutions-without-fixpoints would be universal. However we are not going to assume $\mathrm{AC}_2$. Recall that $j(c)$ is a universal involution. In particular, if $\pi$ is an involution without fixpoints there is a permutation-embedding from $\pi$ into $j(c)$, and any such embedding must send $\pi$ into that part of $j(c)$ that consists of pairs not singletons. Think of $j(c)$ as a set of pairs and singletons and let $X$ be the union of all the pairs in $j(c)$. It's easy to check that

There are $|V|$-many sets that are not closed under complementation (*)

so $X$ is the same size as $V$ and *that* part of $j(c)$ can be copied over to $V$ to give us a permutation $\mathfrak{u}$ of $V$ that has no fixed points. And the construction of $\tau$ ensures that $\pi$ above embeds into it. And $\mathfrak{u}$ is definable!

For (*) reflect that $\{x : V \in x \wedge \emptyset \notin x\}$ is a subset of the collection of sets not closed under complementation, so it will suffice to show that it is of size $|V|$. But it's a moiety of a moiety, in the sense that $B(V)$ is a moiety and provably the same size as $V$, and its members fall into one of two pieces depending on whether or not they contain $\emptyset$, and these two pieces are of course the same size as each other and the same size as $V$. ■

Here is another proof. Any permutation-without-fixpoints must embed into $jc$, and it must embed in that part of it that consists of pairs. We persuade oiurselves that the union of those pairs is of size $|V|$, so we copy it over to a partition of $V$ into pairs. This is the universal permutation-without-fixpoints that we desire.

This merits some reflection. Let us reserve the letter '$\mathfrak{u}$' to denote this definable member of this conjugacy class. This $\mathfrak{u}$ gives us a definable partition of $V$ into pairs which is a kind of definable $\epsilon$-object for bad pairs: if there any counterexamples to $\mathrm{AC}_2$ then $\mathfrak{u}$ is one of them. I don't think this is going to help us prove $\mathrm{AC}_2$, but it is quite striking.

[In the medium term we are going to be interested in finding automorphisms thare are not involutions, that have other cycle types. The cycle types we have to consider are actually quite special. Every automorphism is a fixed point for $j$, and that tells us quite a lot about the cycle type. For any $n$, the number of things belonging to $n$-cycles is either $|V|$ or 0. If there is an $n$-cycle and $m|n$ then there is an $m$-cycle. For these purposes every natural number divides the order of an infinite cycle. This give us $\omega+1$ cycle types we have to worry about, one for each cantorian natural and one for the presence of infinite cycles. We are interested in "universal" permutations of these cycle types and not in any other. It would be nice to show that each of these flavours has a "universal" (top) type. There are some details to be nailed down about the cantorian nature of all these cycles but that is for later.

Is there anything analogous one can say about the cycle types of antimorphisms?]

Going back to what i was saying earlier about pipelining on page 32 . . . what we seem to have established is that the permutation $j\tau$ there is universal. So we can say: $\tau \leq j\tau$ whence $j\tau \leq j^2\tau$ and $j^2\tau$ is universal too. So $j\tau$ and $j^2\tau$ are conjugate. So every model of NF has a permutation model containing an automorphism of infinite order. It's probably universal too (since being universal is a stratified property)

Observe that we are skating here on some pretty thin ice. It might look as if AxCount follows from the existence of an automorphism of inifinite order but it doesn't. Every finite power of an automorphism so there are infinitely many automorphisms and any set of automorphisms is strongly cantorian...? Not so fast!

However it does establish that NF does not prove that the collection of $\in$-automorphisms is a set. Suppose *per contra* that NF proved that the collection of $\in$-automorphisms is a set. We add an automorphism of infinite order by RB permutations. In the resulting permutation model we fix an automorphism $\sigma$ and consider the set of those powers of $\sigma$ that are automorphisms. There is one for each cantorian natural. This is an infinite strongly cantorian set, whence we can infer AxCount. So NF $\vdash$ AxCount ...which is known to be false. So NF $\not\vdash$ "the collection of automorphisms is a set". However we do not know of any permutation $\pi$ for which we can prove $V^\pi \models$ there is no set of all automorphisms"; nor do we know a permutation that will reliably give us a model that doesn't contain an automorphism.

# 9 Working towards Antimorphisms

A bit of a jumble

Are conjugate antimorphisms skew-congugate? Are skew-conjugate antimorphisms conjugate? Is anything conjugate to an antimorphism an antimorphism? Is anything skew-conjugate to an antimorphism an antimorphism?

First let's try to use the idea of $\mathfrak{u}$ the universal permutation-without-fixpoints to get an antimorphism. Suppose $j\mathfrak{u} \cdot c$ were also a universal permutation-without-fixpoints. If it were, then $\mathfrak{u}$ and $j\mathfrak{u} \cdot c$ would be conjugate and then we'd get our antimorphism in a permutation model. If we are to show that $j\mathfrak{u} \cdot c$ is a universal permutation-without-fixpoints it will suffice to show that it has no fixpoints and that $\mathfrak{u} \leq j\mathfrak{u} \cdot c$. In fact it would suffice to show that $jc \leq j\mathfrak{u} \cdot c$

Nathan Bowler has some cold water to pour on this idea. Notice that since we are assuming $\neg\mathrm{AC}_2$ there will be involutions-without-fixpoints that lack transversals so since $\mathfrak{u}$ is universal-without-fixpoints it can have no transversal; so, if $j\mathfrak{u} \cdot c$ is to be conjugate to $\mathfrak{u}$ it had better not have a transversal either. (And even that is a necessary condition not a suff condition). So let us assume $\neg\mathrm{AC}_2$.

We will show that any wellordered subset of $\mathfrak{u}$ without a transversal will give rise to a transversal for $j\mathfrak{u} \cdot c$. So suppose $P = \langle p_i : i < \alpha \rangle$ is a wellordered subset of $\mathfrak{u}$ without a transversal. We will show that $P$ gives a transversal for $j\mathfrak{u} \cdot c$. Let $\{A, B\}$ be a pair in $j\mathfrak{u} \cdot c$; we will show how to choose one of $A$ and $B$. It cannot be that every $p_i$ meets $A$ on a singleton, lest $A$ give rise to a transversal for $P$. $p_i = \{x, \mathfrak{u}(x)\}$ for some $x$ so $p_i$ cannot meet both $A$ and $B = V \setminus \mathfrak{u}``A$. So if $p_i \cap A$ is not a singleton then $p_i \subseteq A$ or $p_i \subseteq B = V \setminus \mathfrak{u}``A$ – but not both! So at least one $p \in P$ must $\subseteq$ precisely one of $\{A, B\}$. There will be a first such $p$, and we can use it to distinguish between $A$ and $B$, and this of course gives us a transversal for $j\mathfrak{u} \cdot c$. If there is to be no transversal for $j\mathfrak{u} \cdot c$ then every wellordered subset of $\mathfrak{u}$ must have a transversal.

■

That's striking, but it's not helpful – it's the wrong way round. The *converse* would say that AC for wellorderable sets of pairs would imply that $j\mathfrak{u} \cdot c$ has no transversal and then we'd be in with a chance of making it conjugate to $\mathfrak{u}$.

In fact this seems entirely general: if $\pi$ is an involution-without-fixpoints and $j\pi \cdot c$ has no transversal then every wellordered subset of $\pi$ has a transversal. That sounds very like a choice principle: something that might not have a transversal does...

I think this observation of Nathan has legs. Watch my back....

Assume that there is a polarity. Let $P$ be a wellordered disjoint family of pairs. We want to show that it has a selection set, a transversal.

There is an injection $f : \bigcup P \hookrightarrow B(\emptyset)$, so we can split $V$ into the two pieces $f``B(\emptyset)$ (which has a partition into pairs) and $(B``\emptyset \setminus f``B(\emptyset)) \cup \overline{B(\emptyset)}$ (which is the same size as $V$ and therefore can be partitioned into pairs) so we obtain a partition of $V$ into pairs which contains a conjugate copy of $P$. Call this partition $\pi$. It's an involution without fixed points, so $\pi \leq \mathfrak{u}$. So every subset of $\pi$ will be conjugate to a subset of $\mathfrak{u}$ and every *wellordered* subset of $\pi$ will be conjugate to a *wellordered* subset of $\mathfrak{u}$. Two conjugate sets of pairs either both have transversals or neither do, so all we need now is that every wellordered subset of $\mathfrak{u}$ should have a transversal. This will follow if $j\mathfrak{u} \cdot c$ has no transversal ... which we had better now prove.

Suppose $\sigma$ is a polarity. It has no fixed points, so $\sigma \leq \mathfrak{u}$ (by universality of $\mathfrak{u}$) whence $j\sigma \cdot c \leq j\mathfrak{u} \cdot c$ (by remark 9 part (ii)). Now $j\sigma \cdot c$ (which is $\sigma$) is a polarity and so has no transversal. So $j\mathfrak{u} \cdot c$ cannot have a transversal. So every wellordered subset of $\mathfrak{u}$ has a transversal, by Bowler's *aperçu* . But $P$ is conjugate to such a wellordered subset and therefore has a transversal.

## 9.1 Antimorphisms have no odd cycles

We start with the observation that no antimorphism can have any odd cycles. One might think this is obvious but it isn't. Things are complicated by the fact that it's far from obvious that a cycle must be cantorian.

**REMARK 11** *No antimorphism can have an odd cycle.*

*Proof:*

What one *wants* to say is this: suppose $\tau$ is an antimorphism and $x$ belongs to a $(2k+1)$−cycle. One then has

$$x \in x \longleftrightarrow \tau^{2k+1}(x) \notin \tau^{2k+1}(x) \tag{A}$$

so we cannot have $\tau^{2k+1}(x) = x$. Unfortunately the biconditional one obtains is not (A), but instead is

$$x \in x \longleftrightarrow \tau^{2k+1}(x) \notin \tau^{2Tk+1}(x).$$

However one can do the following. Suppose $x$ belongs to an $o$-cycle ('$o$' for '$o$dd'). We seek an odd natural number $k$ such that $k$ and $Tk$ are both divisible by $o$. We then have $x \in x \longleftrightarrow \tau^k(x) \notin \tau^{Tk}(x)$. Now, since $k$ and $Tk$ are both divisible by $o$, we have $\tau^k(x)$ and $\tau^{Tk}(x)$ both equal to $x$, whence $x \in x \longleftrightarrow x \notin x$ and we have the contradiction we desired.

Q: But what is $k$?

A: LCM$(o, T^{-1}o)$.

Observe that $o$ divides LCM$(o, T^{-1}o) = k$, and evidently $To$ divides LCM$(To, o) = Tk$. $To$ and $o$ are both odd, and the LCM of two odd numbers is odd.

■

Two points:
(i) Notice that we have not assumed that $\tau$ is a set; so this holds for external antimorphisms as well.
(ii) The proof we have given was complicated by the need to allow for noncantorian cycles. We do at least know that the order of any $\in$-automorphism is cantorian and every set of $\in$-automorphisms is stcan.
Ad (ii). Consider the inductively defined set obtained by closing $\{x\}$ under $\pi$, and the inductive defined set obtained by closing $\{\iota``x\}$ under $j\pi \cdot c$; it seems clear that the size of the second is $T$ of the size of the first. That should show that the size of any $\tau$-cycle ($\tau$ an antimorphism) is cantorian.

Recalling that one of the aims of this investigation is to understand antimorphisms we remind ourselves that $\tau$ is an antimorphism iff $\tau = j(\tau) \cdot c$. This fact gives us an interest in permutations of the form $j\tau \cdot c$ and, in particular, in how the cycle type of $\tau$ controls the cycle type of $j\tau \cdot c$. It's quite easy to see how the cycle type of $\tau$ controls the cycle type of $j\tau$. We remark (without proof for the moment):

if $\tau$ has an $n$-cycle, $j\tau$ has a $Tn$-cycle;
if $\tau$ has infinite cycles, $j\tau$ has cycles of all sizes;
if $\tau$ has cycles of arbitrarily large finite sizes, then $j\tau$ has infinite cycles;
if all $\tau$-cycles have lengths in $I \subset \mathbb{N}$ with $I$ finite then, for $n$ large enough, $j^n\tau$ has cycles of all sizes that divide $LCM(I)$. (With a few '$T$'s scattered around). It also has the maximal possible number of such cycles (presumably $T|V|$) so $j^n\tau$ is going to be universal for $n$ bigger than about 3.

We start the task of seeing how many $n$-cycles $j\tau$ has by consider the simples possible case: how many 2-cycles are ther for $j\tau$ where $\tau$ is the transposition $(a, b)$? Whenever we have $x$ that contains neither $a$ nor $b$ we have a 2-cycle that consists of $x \cup \{a\}$ and $x \cup \{b\}$. There are $|V|$-many such $x$, so precisely $|V|$-many such cycles. Clearly a similar argument will work when $\tau$ has a longer finite cycle. Infinite cycles need a bit more work.

How many fixed points does $j\tau$ have? Clearly $j$ of a transposition $(a, b)$ has $|V|$-many fixed points (every subset of $V \setminus \{a, b\}$ is fixed). If $\tau$ is a bad involution, how many fixed points hath $j\tau$?

$\ldots j\tau \cdot c$ is a lot harder. Lemma 5 is a taster.

**LEMMA 5**

*$AC_2$ implies that, for all permutations $\tau$,*
*$j\tau \cdot c$ has fixed points iff $\tau$ has no odd cycles.*

*Proof:*

R $\to$ L

Suppose $X$ is a fixed point for $j\tau \cdot c$. Then, for each $\tau$-cycle $C$, we must have $\tau``(X \cap C) = C \setminus X$ – and that means that $|C|$ must be even (or infinite).
This direction does not need $\mathrm{AC}_2$.

L $\to$ R

This direction needs $\mathrm{AC}_2$. Suppose $\tau$ has no odd cycles. Each $\tau$-cycle splits into precisely two $\tau^2$ cycles. Use $\mathrm{AC}_2$ to pick, for each $\tau$-cycle, one of the two $\tau^2$-cycles into which it splits. The union of the set of chosen $\tau^2$-cycles is a fixed point for $j\tau \cdot c$.

■

The converse is true too. Suppose $\tau$ is a permutation with no odd cycles, and assume the consequent. Then $j\tau \cdot c$ has a fixed point. $\tau$ itself of course has no fixed point. The fixed point for $j\tau \cdot c$ is a transversal for $\tau$!

Another fairly easy observation in the thread of cycle-type-of-$\tau$-controlling-cycle-type-of-$j\tau \cdot c$ is that...

**REMARK 12**

*(1) If $\tau$ is of order $2n$ then $j\tau \cdot c$ is of order $T2n$;*
*(2) If $\tau$ is of order $2n + 1$ then $j\tau \cdot c$ is of order $T4n + 2$;*
*(3) If $\tau$ has a $\mathbb{Z}$-cycle then so does $j\tau \cdot c$.*

*Proof:*

It's obvious that if $\tau$ is of order $n$ then $j\tau$ is of order $Tn$, but composing with $c$ embroils us in slightly more work.

(1) Suppose $\tau$ is of order $2n$. $c$ commutes with $j\tau$, so in $(j\tau \cdot c)^{T2n}$ we can rearrange to make all the $c$s adjacent and all the $j\tau$ adjacent so they all cancel.

(2) What if the order of $\tau$ is odd? A similar calculation shows that if $\tau$ is of order $2n+1$ then $(j\tau \cdot c)^{T2n+1}$ becomes, with rearrangement-followed-by-cancellation, $(j\tau)^{T2n+1} \cdot c = j(\tau^{2n+1}) \cdot c = \mathbb{1} \cdot c = c$. This is not the identity! However, its square is.

What this shows is the the order **divides** $4n + 2$ not that it **is** $4n + 2$.

(3) Let $x$ belong to a $\tau$ $\mathbb{Z}$-cycle and consider $\{x\}$.

■

Of some interest will be the sequence of permutations:

$$\mathbb{1}, \quad c, \quad jc \cdot c, \quad j^2c \cdot jc \cdot c \ldots,$$

where $c$ is the complementation permutation. The superscripts are all small (they are all concrete numerals, in fact), so – rather than persist with the more general but slightly unwieldy $H(c, i)$ notation of [5] introduced above – we will revert to the simpler (original) notation of Henson, in which these permutations are written '$c_i$', thus: $c_1 := c$; $c_{i+1} := j(c_i) \cdot c$.

Suppose $\mathrm{AC}_2$ fails, so that there are bad involutions with neither fixed points nor transversals. If $\tau$ is a bad involution then, by remark 14, $j\tau \cdot c$ has no fixed points. (It might have a transversal and not be bad...). And, if there are bad involutions, then any involution that is maximal among involutions without fixed points will be bad.

What we might be able to do is this: If $\mathrm{AC}_2$ fails then there is a bad involution, so any involution that is universal for involutions-without-fixpoints (uiwf) is bad. So all we need to show is that if $\tau$ is uiwf then $j\tau \cdot c$ is uiwf.

I think $\tau \mapsto j\tau \cdot c$ is $\leq_B$-monotone.

Sse $\pi\sigma = \tau\pi$. Then $j\pi j\sigma = j\tau j\pi$ and $j(\pi)j(\sigma) \cdot c = j(\tau)j(\pi) \cdot c$ and $j(\pi)j(\sigma) \cdot c \cdot (j\sigma) = j(\tau) \cdot c \cdot j(\pi)$.

However i don't see any reason why it should be injective

If $\tau$ has no fixed points and no transversals then $j\tau \cdot c$ has no fixed points (any fixed point would be a transversal for $\tau$). It is true that there doesn't seem to be anything to prevent $c \cdot j\tau$ having a transversal, so there is work to do. The point of all this, of course, is that if there is a uiwf $\tau$ s.t. $c \cdot j\tau$ is also uiwf then we get a permutation model containing an antimorphism. It a classic fixed-point-for-a-tro-obtained-by-permutations–situation.

Recall that we use lower-case $\mathfrak{fraktur}$ characters for variables ranging over conjugacy classes.

Consider the poset $\langle \mathfrak{P}, \leq_B \rangle$ of conjugacy classes of involutions-without-fixpoints. It is closed under $j$ (which is order-preserving) and $\sigma \mapsto j\sigma \cdot c$ (which isn't). It has a top element which is the conjugacy class of universal involutions; call it $\mathfrak{c}_1$. There are also
(i) the conjugacy class – call it $\mathfrak{c}_3$ – of universal involutions-without-fixpoints, and
(iii) the conjugacy class – call it $\mathfrak{c}_2$ – of the involutions that have a transversal.

Evidently $\mathfrak{c}_2 \leq_B \mathfrak{c}_3$. $\mathrm{AC}_2$ is simply the assertion that $\mathfrak{c}_3 = \mathfrak{c}_2$. And if they are the same then there is only one conjugacy class of involutions-without-transversals. And that's an iff. What happens if $\mathrm{AC}_2$ fails? Then there is more than one conjugacy class. Can we prove that $\mathfrak{c}_2$ is always the bottom element?

If there are involutions-without-fixpoints that have fewer than $T|V|$ pairs (and there might be, for all I know) then the answer would be: no!

If $AC_2$ fails then the congruence class $\mathfrak{c}_1$ of universal involutions consists of bad involutions, and there is the $\leq_B$-minimal class $\mathfrak{c}_2$ which consists of good permutations. In fact it's the equivalence class of $c_1$ – the equivalence class of all $c$s with odd subscripts, indeed. And it's a consequence of corollary 3 that the conjugacy class $\mathfrak{c}_1$ of universal involutions contains all the $c$s with *even* subscripts. $\pi \mapsto j\pi \cdot c$ swaps you back and forth between these two conjugacy classes. (This is how we know that $\pi \mapsto j\pi \cdot c$ does not preserve $\leq_B$). Thus, among the conjugacy classes of involutions we find the conjugacy class of the $c_{2n}$ (which is maximum) and the conjugacy class of the $c_{2n+1}$ (which is minimal). $\mathfrak{c}_2$ is $\leq_B$-mini*mal* but perhaps not minim*um*, since – for all we know – if $AC_2$ fails there may be involutions without fixpoints whose corresponding partitions are *smaller* than $\iota$"$V$. Indeed, to the best of my knowledge, no-one has ever proved that $V$ is not the union of a wellordered(!) family of finite sets. So we should not expect to be able easily to exclude the possibility of partitioning $V$ into fewer than $T|V|$ pairs.

$\langle \mathfrak{P}, \leq_B \rangle$ admits a + operation, arising from disjoint union. Is it the join in the sense of the poset? Well, it will be if $|V|$ is indecomposable. But is it? What happens if $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$?

Consider the class **BINV** of those involutions that are universal or lack fixed points. **BINV** is closed under $\sigma \mapsto j\sigma \cdot c$, which makes it the correct place to search for fixpoints for $\sigma \mapsto j\sigma \cdot c$. We need a name for this function whose fixed points are antimorphisms. Is **BINV** the correct thing to examine? Or its conjugacy classes? Or perhaps its conjugacy classes in $J_1$?? The point being that if $j\sigma \cdot c$ and $j\tau \cdot c$ are $J_1$-conjugate then $\sigma$ and $\tau$ are $J_0$-conjugate.

In fact this setting seems to be one in which various old festering problems appear and can perhaps be partially processed. Among the bad involutions are there any which have fewer than $T|V|$ pairs? Call such a permutation *small bad*. If there are any, is the collection of conjugacy classes of small bad permutations closed under +? This is related to the question of whether or not $|V|$ is decomposable. Clearly if $\sigma$ and $\tau$ are both bad, so is $\sigma \sqcup \tau$. But if $|V|$ is indecomposable, then if $|\sigma \sqcup \tau| = T|V|$, one of $\sigma$ and $\tau$ must also be of size $T|V|$. So might there be a universal small bad permutation?

Let us write '$J_0$' for the symmetric group on $V$, and $J_1$ for $j$"$J_0$ (and so on). Thus the triviality is that $c$ is in the centraliser $C_{J_0}(J_1)$ of $J_1$ in $J_0$. There is slightly more one can say about this that may be worth recording here.

**Remark 13**

$$C_{J_0}(J_1) \;\subseteq\; \{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\} \;\subseteq\; C_{J_0}(\{c, 1\!\!1_V\})^8.$$

*Proof*: First inclusion:

Suppose $\sigma \in C_{J_0}(J_1)$. Let $\tau$ be any permutation whatever. Then

$$\tau``\sigma(x) = \sigma(x)$$

iff (commutativity)

$$\sigma(\tau``x) = \sigma(x)$$

iff (because $\sigma$ is a permutation)

$$\tau``x = x$$

So $\tau$ fixes $\sigma(x)$ setwise iff it fixes $x$ setwise. But $\tau$ was arbitrary. It follows easily[9] that $\sigma(x)$ must be $x$ or $V \setminus x$.

Second inclusion:

Assume $(\forall x)(\pi(x) = x \vee \pi(x) = c(x))$.
We will show that $\pi \in C_{J_0}(\{c, 1\!\!1_V\})$.

If $\pi(x) = c(x)$ then $\pi \cdot c(x) = x$ so $c \cdot \pi(x) = \pi \cdot c(x) = x$.

If $\pi(x) = x$ then $\pi \cdot c(x) = c(x)$ so $c \cdot \pi(x) = c(x) = \pi \cdot c(x)$.

■

Both these inclusions are proper:
$\prod_{x \in \iota``V} (x, V \setminus x)$ is a counterexample to the converse of the first inclusion.
The second inclusion cannot be reversed because $J_1 \subseteq C_{J_0}(\{c, 1\!\!1_V\})$.
Observe that

$$(\{c, 1\!\!1_V\} \subseteq \{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\}$$

whence (because the centraliser function is antimonotonic)

$$C_{J_0}(\{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\}) \subseteq C(\{c, 1\!\!1_V\}).$$

I think I can prove that

$$C_{J_0}(\{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\}) = \{c, 1\!\!1_V\}.$$

The R-to-L inclusion is obvious.

Suppose $a \neq b \neq (V \setminus a)$ and suppose $\tau$ sends $a$ to $b$. Then it doesn't commute with the transposition $(a, V \setminus a)$, which is certainly in $\{\sigma : (\forall x)(\sigma(x) =$

$x \vee \sigma(x) = V \setminus x)\}$ so $\tau$ is not in the centraliser $C_{J_0}(\{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\})$

Not sure what that proves!

$\{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\}$ is actually a group, and of exponent 2, so of course it is abelian and equal to its own centraliser.

Much of what we say below about $c$ goes for any member of $C_{J_0}(J_1)$.

Note that $(\exists\sigma)(y = \sigma``x)$ is an equivalence relation. Let us write it $\sim_1$, and let us write the equivalence class of $x$ under $\sim_1$ (the orbit of $x$ under $J_1$) as $[x]$. What we have shown is that, for each $\pi \in C_{J_0}(J_1)$ and for each $x$, $\pi$ must either fix all members of $[x]$ or send them all to their complements. That is, we can code members of $C_{J_0}(J_1)$ by the equivalence classes whose members they fix. If we now identify $[x]$ and $[V \setminus x]$ by $\approx$ we see that $C_{J_0}(J_1)$ is precisely the additive part of the boolean ring on $(V/{\sim_1})/{\approx}$.

**Lemma 6**
*(i) All the $c_i$ are involutions;*
*(ii) All the $c_i$ commute with each other.*

*Proof:*

(i) We prove this by induction on $i$. Suppose $c_i$ is an involution. $c_{i+1} = jc_i \cdot c$. So $(c_{i+1})^2 = (jc_i \cdot c)^2 = jc_i \cdot c \cdot jc_i \cdot c$. Now by the key triviality we can rearrange to $jc_i \cdot jc_i \cdot c \cdot c = \mathbb{1}$.

In fact this even shows that all products of the $c_i$ are involutions.

(ii) We prove by induction on $i$ that, for all $j$, $c_i$ commutes with $c_j$.

Case $i = 0$. $c_0 = c$ and $c$ commutes with $j(\pi)$ for all $\pi$. But every $c_j$ is $j(\pi) \cdot c$ for some $\pi$, and (compose with $c$ on the right) $j(\pi) \cdot c \cdot c = j(\pi)$ and if we compose with $c$ on the left we get $c \cdot j(\pi) \cdot c$ which, too, is $j(\pi)$ because $c$ commutes with $j(\pi)$.

Now for the induction.

$$c_{i+1} \cdot c_j = j(c_i) \cdot c \ \cdot \ j(c_{j-1}) \cdot c$$

and the RHS simplifies to

$$j(c_i) \cdot j(c_{j-1})$$

which is

$$j(c_i \cdot c_{j-1})$$

which by induction hypothesis is

$$j(c_{j-1} \cdot c_i)$$

[8] Actually one can spice this up quite a lot, by reflecting that the centraliser function is antimonotonic, so one can whack '$C()$' in front of each of these and then reverse all the arrows. I was sure I had written this out somewhere but I can't find it.

[9] I suspect the proof that I am eliding is not constructively correct.

which is

$$j(c_{j-1}) \cdot j(c_i).$$

We now sprinkle a couple of $c$s judiciously – by the triviality we know can insert them anywhere – obtaining

$$j(c_{j-1}) \cdot c \ \cdot \ j(c_i) \cdot c$$

which is of course

$$c_j \cdot c_{i+1}.$$

■

**Remark 14**
*Let $\sigma$ and $\tau$ be involutions of $V$.*

*(1) Let $\tau$ be an involution without fixpoints. Then $\mathcal{T}$ is a transversal for $\tau$ iff $\mathcal{T}$ is a fixpoint for $j\tau \cdot c$;*

*(2) $\mathcal{T}$ is a fixpoint for $\sigma$ iff $B(\mathcal{T})$ is a transversal for $j\sigma \cdot c$.*

*Proof:*
(1) Think of $\tau$ as a partition of $V$ into pairs. Then, if $\mathcal{T}$ is a transversal, $V \setminus \mathcal{T}$ (which is also a transversal) is precisely $\tau``\mathcal{T}$.

(2) A piece of [the partition] $j\sigma \cdot c$ is a pair $\{x, V \setminus \sigma``x\}$ – which of course might be a singleton. If $\sigma(T) = T$ then, for all $x$, precisely one of $x$ and $V \setminus \sigma``x$ will contain $T$. That is to say, $\{x, V \setminus \sigma``x\} \cap B(\mathcal{T})$ is a singleton, so $B(\mathcal{T})$ is a transversal.

For the other direction …if $B(\mathcal{T})$ is a transversal for $j\sigma \cdot c$ then, for all $x$, precisely one of $x$ and $V \setminus \sigma``x$ contains $T$, which is to say that $\mathcal{T} \in x \longleftrightarrow \sigma(\mathcal{T}) \in x$. In particular let $x$ be $\{\mathcal{T}\}$; then $\mathcal{T} \in \{\mathcal{T}\} \longleftrightarrow \sigma(\mathcal{T}) \in \{\mathcal{T}\}$, so $\sigma(\mathcal{T}) = \mathcal{T}$.

■

I thought this corollary followed but it doesn't.

Error Alert! This is not a corollary

**Corollary 2**
*If $j\tau \cdot c$ is bad then $\tau$ is bad.*

*Proof:*
Suppose $j\tau \cdot c$ is bad. Then it has no transversals. In particular for no $\mathcal{T}$ is $B(\mathcal{T})$ a transversal, so for no $\mathcal{T}$ is $\mathcal{T}$ a fixpoint for $\tau$.
Suppose $j\tau \cdot c$ is bad. Then it has no fixpoints. So $\tau$ has no transversals. ■

At one point i tho'rt i had a proof of the convers, but i didn't.
We aspire to show $\tau$ bad implies $j\tau \cdot c$ bad. (which won't work!)
$\tau$ has no fixpoint so $j\tau \cdot c$ has no transversal[10].

[10]No! It might have transversals that aren't $B$ of any fixpoint for $\tau$

$\tau$ has no transversal. Suppose, *per impossibile*, that $j\tau \cdot c$ has a fixpoint, $x$. Then $x = V \setminus \tau``x$ which says that $x$ is a transversal for $\tau$.

■

The gap could be plugged if there were a way of constructing a fixpoint for an involution $\tau$ from a transversal for $j\tau \cdot c$.

The following corollary seems quite striking, but it hasn't borne any fruit just yet.

**COROLLARY 3**
*(i) For any ultrafilter $\mathcal{U}$ on $V$, $B^n(\mathcal{U})$ is a transversal for $c_{2n+1}$;*
*(ii) All the $c_{2n+1}$ are conjugate;*
*(iii) For all $n \geq 1$, $c_n$ is conjugate to $c_{n+2}$.*

*Proof:*
(i) We do an induction on $n$.

For the case $n = 0$ any ultrafilter is a transversal for $c$.
Suppose for the induction that $B^{n-1}(\mathcal{U})$ is a transversal for $c_{2n-1}$.
Consider

$$c_{2n+1}(A) \in B^n(\mathcal{U}). \qquad ((\mathrm{A}))$$

By definition of $B$ this is the same as

$$B^{n-1}(\mathcal{U}) \in c_{2n+1}(A)$$

Now $c_{2n+1}(A) = V \setminus (c_{2n}``A)$, so we can rewrite (A) as

$$c_{2n}(B^{n-1}(\mathcal{U})) \notin A.$$

By induction hypothesis $B^{n-1}(\mathcal{U})$ is a transversal for $c_{2n-1}$, which is to say that $B^{n-1}(\mathcal{U})$ is a fixed point for $c_{2n}$. So rewrite '$c_{2n}(B^{n-1}(\mathcal{U}))$' as '$B^{n-1}(\mathcal{U})$'; this turns our formula-in-hand into

$$B^{n-1}(\mathcal{U}) \notin A$$

which (by definition of $B$) becomes

$$A \notin B^n(\mathcal{U}).$$

So we have proved

$$c_{2n+1}(A) \in B^n(\mathcal{U}) \longleftrightarrow A \notin B^n(\mathcal{U})$$

...which is to say that $B^n(\mathcal{U})$ is a transversal for $c_{2n+1}$.

(ii) now follows by lemma 2.

(iii) By induction on $n$.

The case $n = 1$ we know from (ii).

For the induction step suppose $\pi$ conjugates $c_n$ to $c_{n+2}$, which is to say

$$\pi \cdot c_n \cdot \pi^{-1} = c_{n+2}$$

Lift by $j$:

$$j\pi \cdot j(c_n) \cdot (j\pi)^{-1} = j(c_{n+2})$$

compose both sides with $c$ on the right:

$$j\pi \cdot j(c_n) \cdot (j\pi)^{-1} \cdot c = j(c_{n+2}) \cdot c$$

But $c$ commutes with $(j\pi)^{-1}$ so we can rearrange the LHS, and $j(c_{n+2})\cdot c = c_{n+3}$ on the RHS giving

$$j\pi \cdot \underline{j(c_n) \cdot c} \cdot (j\pi)^{-1} = c_{n+3}$$

Now $j(c_n) \cdot c$ (underlined) $= c_{n+1}$ giving

$$j\pi \cdot c_{n+1} \cdot (j\pi)^{-1} = c_{n+3}$$

as desired.

■

Also worth minuting is the fact that

**REMARK 15** *Conjugacy is a congruence relation for the operation $\pi \mapsto j\pi \cdot c$.*

*Proof:*
Suppose $\sigma$ and $\tau$ are conjugate; so, for some $\pi$,

| | |
|---|---|
| $\pi \cdot \sigma \cdot \pi^{-1} = \tau$; | Then whack it with $j$: |
| $j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) = j(\tau)$; | compose with $c$: |
| $j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) \cdot c = j(\tau) \cdot c$; | but $c$ commutes with $j$ of anything, giving: |
| $j(\pi) \cdot j(\sigma) \cdot c \cdot j(\pi^{-1}) = j(\tau) \cdot c$ | |

which says that $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ are conjugated by $j(\pi)$.

■

Notice that in this construction $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ end up being conjugated by $j$ of something, which is (presumably, demonstrably?) a stronger condition than simply being conjugate. There seems to be no obvious reason why the induced function $[\sigma] \mapsto [j\sigma \cdot c]$ on conjugacy classes should be injective.

I think we prove it isn't

**LEMMA 7** *(Bowler, written up by tf)*
*$j(c)$ and $c_2$ are conjugate, so $c_2$, too, is universal.*

*Proof:*

duplication. . . ??

Given a set of the form $x \triangle B(\emptyset)$ we can recover $x$ since it is $(x \triangle B(\emptyset)) \triangle B(\emptyset)$. So $x \mapsto x \triangle B(\emptyset)$ is injective. But the same thought reassures us that it is surjective too, so it is genuinely a permutation of $V$ and, actually, an involution. In fact we can write it $\prod_{x \in V}(x, x \triangle B(\emptyset))$ as a product of disjoint transpositions . . . or $\pi$ for short.

To see that $\pi$ conjugates $c_2$ to $j(c)$, we calculate as follows:

$$\begin{aligned}
(j(c)\cdot\pi)(x) &=^1 j(c)(x \mathbin{\triangle} B(\emptyset))\\
&=^2 j(c)(x) \mathbin{\triangle} j(c)(B(\emptyset))\\
&=^3 j(c)(x) \mathbin{\triangle} (V \setminus B(\emptyset))\\
&=^4 j(c)(x) \mathbin{\triangle} (V \mathbin{\triangle} B(\emptyset))\\
&=^5 (j(c)(x) \mathbin{\triangle} V) \mathbin{\triangle} B(\emptyset)\\
&=^6 (V \setminus j(c)(x)) \mathbin{\triangle} B(\emptyset)\\
&=^7 (c \cdot j(c))(x) \mathbin{\triangle} B(\emptyset)\\
&=^8 c_2(x) \mathbin{\triangle} B(\emptyset)\\
&=^9 (\pi \cdot c_2)(x)
\end{aligned}$$

Equation 1 holds by definition of $\pi$;
Equation 2 distribute $j$ over $\triangle$;
Equation 3 holds beco's $jc$ swaps ultrafilters with their complements;
Equation 4 holds beco's $V \setminus X = V \mathbin{\triangle} X$;
Equation 5 holds beco's $\triangle$ is associative;
Equation 6 holds beco's $V \setminus X = V \mathbin{\triangle} X$;
Equation 8 holds by definition of $c_2$ (It equals $jc \cdot c$);
Equation 9 holds by definition of $\pi$.
■

> *How could he see that???!* There are two facts that seem to be key. One is that $V \triangle X = V \setminus X$, and the other is that $jc$ swaps ultrafilters with their complements, and so does the same as $c$ in that case. Perhaps these two insights can be put to wider use.

**Corollary 4**
*Every model of NF has a permutation model with an internal $\in$-automorphism.*

*Proof:* It follows from corollary 1 that $j(c)$ and $j^2(c)$ are conjugate, making $j(c)$ an example of a permutation which is conjugate to $j$ of itself. It was shown in [8] that any model containing such a permutation $\pi$ has a permutation model wherein $\pi$ has become an (internal) $\in$-automorphism. ■

In [8] it is shown that there must be such a $\pi$, but that was on the assumption of $AC_2$, and of course we have here scrupulously eschewed $AC_2$.

Zuhair Abdul Ghafoor Al-Johar has asked me whether the automorphism obtained in this way moves any wellfounded set. Thinking about it for a bit the answer is of course 'no'. For any automorphism $\sigma$ the set $\{x : \sigma(x) = x\}$ is indeed a set and it extends its own power set, so – by induction – it contains all wellfounded sets.

For the main result which follows later (corollary 6) we will need involutions $\sigma$ and $\tau$ such that there is a permutation $\pi$ conjugating $\sigma$ to $j(\tau) \cdot c$ and $\tau$ to $j(\sigma) \cdot c$. The next lemma exhibits such a pair of involutions, taking $\sigma$ to be $c_1$ and $\tau$ to be $c_2$.

**Lemma 8** *(Bowler)*

*There is an involution that conjugates $c$ with $c_3$ and commutes with $c_2$.*

We seem to have accumulated two proofs of this

*Proof:*

We begin by choosing a fixed point $a$ of $c_2$ and setting $b = c_1(a)$. Since $a$ is a fixed point of $c_2$ we also have $b = c_1(c_2(a)) = j(c)(a)$. For any $s \subseteq \{a, b\}$ we define $X_s$ to be $\{x : x \cap \{a, b\} = s\}$.

$X_\emptyset$ is closed under both $j(c)$ and $j^2(c)$; let $\sigma_\emptyset$ be the restriction of $j(c)$ to $X_\emptyset$ and $\tau_\emptyset$ the restriction of $j^2(c)$. Then there are embeddings of $j(c)$ into $\sigma_\emptyset$ and $j^2(c)$ into $\tau_\emptyset$, so by the results of the last section both $\sigma_\emptyset$ and $\tau_\emptyset$ are universal. Let $\pi_\emptyset$ be an isomorphism from $\sigma_\emptyset$ to $\tau_\emptyset$. Since $j(c) = c_1 \cdot c_2$ and $j^2(c) = c_3 \cdot c_2$ we have the equation $\pi_1 \cdot c_1 \cdot c_2 = c_3 \cdot c_2 \cdot \pi_1$, which we note for future use.

We now define $\pi : V \to V$ by

$$x \mapsto \begin{cases} \pi_\emptyset(x) & \text{if } x \cap \{a,b\} = \emptyset \\ x & \text{if } x \cap \{a,b\} = \{b\} \\ c_3(c_1(x)) & \text{if } x \cap \{a,b\} = \{a\} \\ c_3(\pi_\emptyset(c_1(x))) & \text{if } x \cap \{a,b\} = \{a,b\} \end{cases}$$

Then $\pi$ is a union of bijections from $X_s$ to $X_s$ for each $s \subseteq \{a, b\}$, so it is a bijection.

It remains to check that for any $x$ we have $\pi(c_1(x)) = c_3(\pi(x))$ and $\pi(c_2(x)) = c_2(\pi(x))$. For each equation there are four cases, depending on $x \cap \{a, b\}$. We now check these cases for the first equation.

- If $x \cap \{a, b\} = \emptyset$, then $c_1(x) \cap \{a, b\} = \{a, b\}$ and so

$$\pi(c_1(x)) = c_3(\pi_\emptyset(c_1(c_1(x)))) = c_3(\pi_\emptyset(x)) = c_3(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{b\}$ then $c_1(x) \cap \{a, b\} = \{a\}$ and so

$$\pi(c_1(x)) = c_3(c_1(c_1(x))) = c_3(x) = c_3(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a\}$ then $c_1(x) \cap \{a, b\} = \{b\}$ and so

$$\pi(c_1(x)) = c_1(x) = c_3(c_3(c_1(x))) = c_3(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a, b\}$ then $c_1(x) \cap \{a, b\} = \emptyset$ and so

$$\pi(c_1(x)) = \pi_\emptyset(c_1(x)) = c_3(c_3(\pi_\emptyset(c_1(x)))) = c_3(\pi(x))\,.$$

Might be an idea to annotate these equations

The four cases for the other equation are similar.

- If $x \cap \{a, b\} = \emptyset$ then $c_2(x) \cap \{a, b\} = \{a, b\}$ and so

$$\pi(c_2(x)) = c_3(\pi_\emptyset(c_1(c_2(x)))) = c_3(c_3(c_2(\pi_\emptyset(x)))) = c_2(\pi_\emptyset(x)) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{b\}$ then $c_2(x) \cap \{a, b\} = \{b\}$ and so

$$\pi(c_2(x)) = c_2(x) = c_2(\pi(x))\,.$$

- If $x \cap \{a,b\} = \{a\}$ then $c_2(x) \cap \{a,b\} = \{a\}$ and so

$$\pi(c_2(x)) = c_3(c_1(c_2(x)))) = c_2(c_3(c_1(x))) = c_2(\pi(x))\,.$$

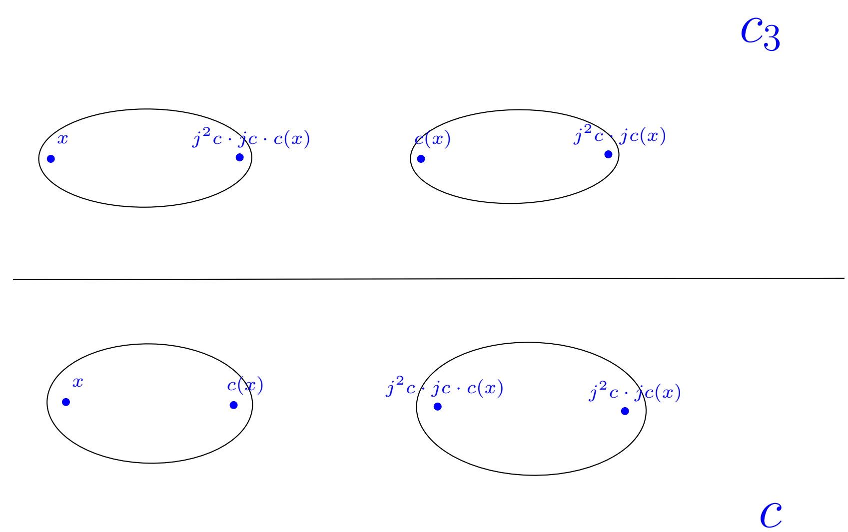
- If $x \cap \{a,b\} = \{a,b\}$ then $c_2(x) \cap \{a,b\} = \emptyset$ and so

$$\pi(c_2(x)) = \pi_\emptyset(c_2(x)) = \pi_\emptyset(c_2(c_1(c_1(x)))) = c_2(c_3(\pi_\emptyset(c_1(x)))) = c_2(\pi(x))\,.$$

$\blacksquare$

Here is another proof

The universe partitions naturally into bundles closed under both $c_1$ and $c_3$. Each such bundle contains precisely four sets. We will define a permutation $\pi$ in such a way that it fixes each bundle setwise. It will turn out that $\pi$ is the permutation we seek.




In the accompanying picture we have written a typical bundle twice: once below the line where it is divided into two $c$-cycles and once above the line where it is divided into two $c_3$-cycles. We have to biject the set of points below the line with the set of points above the line in a way that respects the two partitions into cycles. Evidently this can be done (in eight different ways, as it happens) so we pick one such way for each bundle. By corollary 3 (i) we have transversals for $c_3$ and $c$. The transversal for $c_3$ highlights precisely one element in each pair upstairs, namely that element that contains $B(\emptyset)$. These two highlighted elements cannot – downstairs – belong to different pairs because the downstairs pairs are complements and two complementary sets cannot both contain $B(\emptyset)$.

To illustrate, suppose in the picture that upstairs we highlight $x$ and (therefore) $j^2c \cdot jc(x)$. We tell $\pi$ to fix these two sets, and that compels it to swap $c(x)$ and $c_3(x)$.

The other possibility is that we highlight $c_3(x)$ and $c(x)$, and then we tell $\pi$ to fix those two sets and to swap $x$ and $j^2c \cdot jc(x)$.

Either way the net result is that $\pi$ is

$$\texttt{if } B(\emptyset) \in x \texttt{ then } x \texttt{ else } j^2c \cdot jc(x).$$

Reflect that $B(\emptyset) \in x$ iff $B(\emptyset) \in j^2c \cdot jc(x)$, and $j^2c \cdot c$ is an involution. So, if $B(\emptyset) \in x$, it follows that $\pi(x) = x$ and then $\pi^2(x) = x$; if $B(\emptyset) \notin x$ then $\pi(x) = j^2c{\cdot}jc(x)$ which does not contain $B(x)$ either. So $\pi^2(x) = \pi(j^2c{\cdot}jc(x)) = j^2c \cdot jc \cdot j^2c \cdot (x) = x$ and $\pi^2(x) = x$. So $\pi$ is an involution.

Let us check that $\pi$ commutes with $j^2c \cdot jc$, that is to say: $j^2c \cdot jc \cdot \pi(x) = \pi \cdot j^2c \cdot jc(x)$ for all $x$.

There are two cases, depending on whether or not $B(\emptyset) \in x$.

If $B(\emptyset) \in x$ then $\pi(x) = x$ and $j^2c \cdot jc \cdot \pi(x) = j^2c \cdot jc(x)$.

If $B(\emptyset) \in x$ then $B(\emptyset) \in j^2c \cdot jc(x)$ so $j^2c \cdot jc(x)$ is fixed by $\pi$.

Either way $j^2c \cdot jc \cdot \pi(x) \;=\; \pi \cdot j^2c(x) \;=\; \pi \cdot j^2c \cdot jc(x)$

If $B(\emptyset) \notin x$ then $\pi(x) = j^2c \cdot jc(x)$. Then $j^2c \cdot jc \cdot \pi(x) = x$.

If $B(\emptyset) \notin x$ then $B(\emptyset) \notin j^2c \cdot jc(x)$ so $j^2c \cdot jc(x)$ is moved by $\pi$, and must be $x$.

Either way $j^2c \cdot jc \cdot \pi(x) \;= x \;=\; \pi \cdot j^2c \cdot jc(x)$

■

Presumably there is a generalisation that says that there is an involution that conjugates $c_i$ with $c_{i+2}$ and commutes with $j^{i+2}c{\cdot}j^{i+1}c$. But – presumably – we are not going to need it.

However i think this is completely general. Is it not the case that, in any symmetric group, if $\sigma$ and $\tau$ are conjugate, then they can be conjugated by something that commutes with $\sigma\tau$? Something like that must be true...

## 9.2 Finding Permutations that will prove Duality[2]

**Corollary 5**

*Every model of NF has a permutation model that contains two (internal) permutations $\sigma$ and $\tau$ satisfying*
$(\forall xy)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y))$ *and*
$(\forall xy)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y))$.

*Furthermore any such model satisfies duality for formulæ that are stratifiable-mod-2.*

*Proof:* We use the permutation $\pi$ from lemma 8, and exploit the two permutations $\sigma$ and $\tau$ that we find in the permutation model $V^{\pi}$.

If a formula $\phi$ is stratifiable-mod-2 then its variables can be assigned to two types `yin` and `yang` in such a way that in subformulæ like '$x = y$' the two variables receive the same type and in subformulæ like '$x \in y$' the two variables receive different types. Let us associate $\sigma$ to variables given type `yin` in the assignment and associate $\tau$ to variables given type `yang` in the assignment. '$x \in y$' is equivalent to '$\sigma(x) \notin \tau(y)$' and if $x$ is of type `yin` we perform this rewrite. '$x \in y$' is also equivalent to '$\tau(x) \notin \sigma(y)$' and if $x$ is of type `yang` we perform that rewrite. We deal with equations analogously. In the rewritten version of $\phi$ we find that every variable '$x$' of type `yin` now appears only as '$\sigma(x)$' and that every variable '$y$' of type `yang` now appears only as '$\tau(y)$'. So we can reletter '$\sigma(x)$' as '$x$', and '$\tau(y)$' as '$y$' and the result is $\phi^{\circlearrowleft}$. ■

There is a further corollary: no homogeneous formula $\phi(x_1, x_2)$ can define a BFEXT (a well-founded extensional binary relation) on $V$. Given a definable well-founded extensional binary relation on $V$ we can argue as follows. Let $\sigma$ be an permutation, assumed to be an $\in$-automorphism. We then prove by wellfounded induction on $\phi$ that $\sigma$ is the identity.

Actually we have to be very careful how we state this ...

First we prove that if there is a definable wellfounded extensional relation on the whole of $V$ then there are no nontrivial $\in$-automorphisms.

Suppose $\sigma$ is an $\in$-automorphism, and that $\phi(x, y)$ defines a wellfounded extensional relation on the whole of $V$. Fix $y$ and suppose $(\forall x)(\phi(x, y) \to x = \sigma(x))$. Then $(\forall x)(\phi(x, y) \longleftrightarrow \phi(x, \sigma(y)))$ whence $y = \sigma(y)$ by extensionality. Then if $\{y : y \neq \sigma(y)\}$ is nonempty it has no $\phi$-minimal element, contradicting wellfoundedness of $\phi$.

We plan next to exploit corollary 4. The obvious thing to do is to say: suppose $\phi$ defines a wellfounded extensional binary relation on $V$; jump into a permutation model containing a nontrivial $\in$-automorphism to prove that it's not a wellfounded extensional binary relation. However for that to work we need the expression "$\phi$ defines a wellfounded extensional binary relation on $V$" to be stratified, and for that we need $\phi$ to be stratified. It doesn't have to be homogeneous, but it does have to be stratified.

## Some questions

1. *Under what operations is the class of universal involutions closed?*

2. *Are the universal involutions a normal generating subset of $J_0$?*

*Ad* (2): Every permutation is a product of involutions; is every permutation a product of universal involutions? Surely we can bring Bowler-Forster to bear on this. If not, then the subgroup generated by the universal involutions is a normal proper subgroup of small index. [We'd better show that this normal subgroup is of small index in the meaning of the act!] Come to think of it, one can run the same argument for all conjugacy classes of involutions.

**Corollary 6** *Every model of NF has a permutation model that satisfies duality for formulæ that are stratifiable-mod-2.*

■

It's worth bearing in mind that $\sigma$ and $\tau$ retain in $V^\pi$ all the stratified properties they had in their previous life in $V$, where they were $c$ and $c_2$. Thus they commute, and $\sigma^2 = \tau^2 = \mathbb{1}$. Observe also that

$$j(\sigma\tau) = j\sigma \cdot j\tau = \tau \cdot c \cdot c \cdot \sigma = \tau\sigma = \sigma\tau,$$

so $\sigma\tau$ is actually an $\in$-automorphism of $V^\pi$. It is a nontrivial automorphism beco's $\sigma$ and $\tau$ are not inverse to each other: $\tau$ has fixed points and $\sigma$ does not. By the remark in the proof of part (i) of lemma 6 $\sigma\tau$ is an involution.

This fact is worth recording!

But we proved this aleady on p 32.

**Corollary 7** *Every model of NF has a permutation model containing a nontrivial automorphism of order 2.*

We should be able to express this as a fact inside the base model...

Can we use this technique to obtain models in which duality holds for formulæ that are stratifiable-mod-$p$ for other primes? If we were to rejig the above development to seek a proof for formulæ that are stratifiable-mod-3 then we would be looking for an *antimorphism tuple* (in this case *triple*) $\sigma$, $\tau$, $\pi$ such that there is $\theta$ satisfying

$$(j\sigma \cdot c)^\theta = \tau, \ \ (j\tau \cdot c)^\theta = \pi \text{ and } (j\pi \cdot c)^\theta = \sigma.$$

However, as Nathan Bowler has reminded me, the existence of such a triple contradicts $AC_2$ since – by lemma 5 – $\tau$ has an odd cycle iff $j\tau \cdot c$ does not. And if we are going to ditch $AC_2$ then we may as well go for outright antimorphisms from day 1.

## 9.3 Full Duality?

It may be that the set of things fixed by $\sigma\tau$ is a model of NF + full Duality. Something to check!

First we check that $\sigma\tau$ (which is the same as $\tau\sigma$) is an $\in$-automorphism. For all $x$ and $y$ we have $x \in y \longleftrightarrow \sigma(x) \notin \tau(y)$ so $\sigma(x) \notin \tau(y) \longleftrightarrow \tau\sigma(x) \in \sigma\tau(y) = \tau\sigma(y)$ so $\tau\sigma$ is an $\in$-automorphism as desired.

Next we check that if $\pi$ is an $\in$-automorphism then the set of fixed points is a model of NF. The big gap here is extensionality. We would have to show that every nonempty fixed set has a fixed member.

Finally we check that the set of fixed points of $\sigma\tau$ is additionally a model of duality. Observe that, for all such fixed $x$ we have $x = \sigma(\tau(x))$ whence $\sigma^{-1}(x) = \tau(x)$. But $\sigma^2 = \mathbb{1}$ so $\sigma(x) = \tau(x)$.

Now suppose $x$ and $y$ both fixed. Then $x \in y \longleftrightarrow \sigma(x) \notin \tau(y) = \sigma(y)$. So $\sigma$ is an antimorphism of the fixed points.

But this relies on the set of fixed points being extensional. It may be that we can ensure this by a judicious choice of the permutation in lemma 8. We seek a $\pi$ that conjugates $c$ to $j^2c \cdot jc \cdot c$ and moreover has the extra feature that in $V^\pi$ the set $\{x : \sigma(x) = \tau(x)\}$ is extensional. Must turn this into a condition on $\pi$.... We think

$$V^\pi \models (\forall x)(x \neq \emptyset \wedge \sigma\tau(x) = x \rightarrow (\exists y \in x)(\sigma\tau(y) = y))$$

is

$$(\forall x)(\pi(x) \neq \emptyset \wedge \sigma\tau(x) = x \rightarrow (\exists y \in \pi(x))(\sigma\tau(y) = y))$$

which becomes

$$(\forall x)(x \neq \emptyset \wedge j^2c \cdot jc(x) = x \rightarrow (\exists y \in \pi(x))(j^2c \cdot jc(y) = y))$$

where $\pi$ conjugates $c$ and $j^2c \cdot jc \cdot c$.

Let us write '$F$' for $\{x : x = jc \cdot j^2c(x)\}$ to keep things readable. The $\pi$ we seek has got to inject $F$ into $\{y : y \cap F \neq \emptyset\}$ – o/w known (see p. 7) as "Ƅ$(F)$". Observe that Ƅ$(x)$ is always a moiety, since it is $V \setminus (\mathcal{P}(V \setminus x))$, and the complement of a power set (of anything other than $V$) is always the same size as $V$. This is beco's every set (other than $V$ itself) is included in the complement of a singleton, and the power set of a complement of a singleton is a principal prime ideal and therefore a moiety.

So there's no problem on *that* score.

It's not blindingly obvious to me that it cannot be done.

### 9.4 Refuting duality

The Lads said:

> **First**: Add a Quine atom by $\tau = (\emptyset, \{\emptyset\})$;
> **Second**: Kill off all Quine atoms by $\tau = \prod_{x \in \iota^2 \text{``} V} (x, V \setminus x)$.

Now it should be possible to do it with a single permutation. I think the idea is to swap with their complements-in-the-sense-of-$(\emptyset, \{\emptyset\})$, all those sets that are double singletons in the sense of $V^{(\emptyset, \{\emptyset\})}$. That is to say – writing '$\sigma$' for the transposition $(\emptyset, \{\emptyset\})$ and '$c$' for complementation to keep things readable:

$$\tau := \prod_{(x \in \iota^2 \text{``} V)^\sigma} (x, \sigma c \sigma(x))$$

is the one-stop permutation we want. (The fact that this definition is legitimate is nontrivial: it's a great help that $\sigma c \sigma$ is an involution. We also need the fact that if $x$ is a double-singleton-in-the-sense-of-$\sigma$ then its complement-in-the-sense-of-$\sigma$ cannot be a double-singleton-in-the-sense-of-$\sigma$. This ensures that all the transpositions in the big product are disjoint.)

**THEOREM 1**

*Duality fails in $V^\tau$ because it contains a Quine antiatom but no Quine atom.*

*Proof:*

Clearly the collection $A := \{x : ((\exists z)(x = \{\{z\}\}))^\sigma)\}$ is going to be of interest. Let's process '$(x \in \iota^2 ``V)^\sigma$'.

$$(x \in \iota^2 ``V)^\sigma$$

is

$$(\exists z)(x = \{\{z\}\})^\sigma$$

which is

$$(\exists z)(\sigma``(\sigma(x)) = \{\{z\}\})$$

which is

$$(\exists z)(\sigma(x) = \sigma``\{\{z\}\})$$

which is

$$(\exists z)(\sigma(x) = \{\sigma\{z\}\}).$$

Two things to notice

1. Since every Quine atom is fixed by $\sigma$ every Quine atom belongs to $A$. Everything that starts life as a Quine atom is moved.
2. Notice too that if $x = \emptyset$ then it belongs to $A$: $\sigma(\emptyset) = \{\emptyset\} = \{\sigma\{\emptyset\}\}$.

So what is the fate of $\emptyset$ in the new model $V^\tau$? (Let's call it '$a$' in order not to confuse ourselves!)

$$(x \in a)^\tau$$

iff

$$x \in \tau(a)$$

Now $\tau(a)$ is the complement-in-the-sense-of-$V^\sigma$ of $a$ which is $\sigma c \sigma(a) = \sigma c\{\emptyset\} = \sigma(V \setminus \{\emptyset\}) = V \setminus \{\emptyset\}$.

iff

$$x \in (V \setminus \{\emptyset\})$$

iff

$$x \notin \{\emptyset\}$$

iff

$$x \neq \emptyset$$

iff

$$x \neq a$$

So $a$ is a Quine antiatom in the new model $V^\tau$.

Now let's check that there are no Quine atoms in the new model $V^\tau$.

Suppose $x$ is a Quine atom in the sense of $V^\tau$. If $x$ is fixed by $\tau$ then it was a Quine atom in the model in which we started. We observed earlier (item 2 p 53) that any object that starts life as a Quine atom is moved by $\tau$. So $x$ is moved. So ($x$ is a Quine atom)$^\tau$ is

$$(\forall y)(y = x \longleftrightarrow y \in \sigma c \sigma x)$$

We need not consider the case where $x = \emptyset$, since we have already dealt with that and seen that $x$ is a Quine antiatom. If $x = \{\emptyset\}$ then the RHS becomes $y \in \sigma c \sigma\{\emptyset\} = V$ which is clearly not equivalent to the LHS; clearly $\{\emptyset\}$ is not a Quine atom in $V^\tau$.

There remain the cases where $x$ is fixed by $\sigma$. These give

$$(\forall y)(y = x \longleftrightarrow y \in \sigma(V \setminus x))$$

For $x$ to be a Quine atom in $V^\tau$, $\sigma(V \setminus x)$ will have to be a singleton. This can happen if $x = V$, for then $V \setminus x$ is empty and $\sigma(V \setminus x)$ is $\{\emptyset\}$ so $x$ would have to be both $V$ and $\emptyset$, so the case $x = V$ does not give rise to a Quine atom. The only other way for $\sigma(V \setminus x)$ to be a singleton is for $V \setminus x$ to be a singleton, say $\{z\}$ and for it to be fixed by $\sigma$. In that case the condition for ($x$ is a Quine atom)$^\tau$ becomes

$$(\forall y)(y = x \longleftrightarrow y \in (V \setminus \{z\}))$$

which is clearly impossible. ■

In contrast, we have not yet found a permutation model that satisfies duality.

# 10 Work still to do

The previous section is error-strewn of course! Here is a message from Nathan

"Hi Thomas,

I think I've found a way to recover your original result (given the axiom of choice for pairs, we get a permutation model such that truth of sentences which can be 2-stratified is preserved under complementing the containment relation).

I don't have time to send you more than a sketch now, but I'll try to send more details later. Fix some identification of $V$ with $V \times \{1,2,3,4,5,6,7,8\}$. Let $s$ be the permutation acting on the second component by $(1,2)(3,4)(5,6)(7,8)$ and $t$ the permutation acting on the second component by $(3,4)(5,8)(6,7)$. So $s$ and $t$ are commuting involutions. If we look at the orbits under the action of these two permutations, we get:

- sets of the form $\{x\} \times \{1,2\}$ – here $s$ exchanges the two elements and $t$ fixes both.
- sets of the form $\{x\} \times \{3,4\}$ – here both $s$ and $t$ exchange the two elements.
- sets of the form $\{x\} \times \{5,6,7,8\}$ – here $s$ exchanges $(x,5)$ with $(x,6)$ and $(x,7)$ with $(x,8)$, whereas $t$ exchanges $(x,5)$ with $(x,8)$ and $(x,6)$ with $(x,7)$.

Now consider $s' = j(t) \cdot c$ and $t' = j(s) \cdot c$. The 'orbit type' of the pair $(s', t')$ is the same as that for the pair $(s,t)$. So there is some permutation conjugating $s$ to $s'$ and $t$ to $t'$, which is exactly what we need."

There remains of course the challenge of proving consistency of duality for all sentences, not merely those that are stratifiable-mod-2. But more to the point are the possibilities of extending to formulæ that are stratifiable-mod-$n$ things known about the rather more restricted class of stratified formulæ – and these we haven't started thinking about. Here are some, in no particular order.

- We should show in an NF context that, for each $n$, the assertion that "there are sets $x$ s.t. $\iota^n \restriction x$ exists" is invariant.

- Is there any interest in versions of Forti-Honsell Antifoundation along the lines "Every set picture that is a $n$-stratification graph is a picture of a set"?

- If $\phi$ is, for each $n$, equivalent (modulo NF) to something that is stratified-mod-$n$ must it be (NF)-invariant?

  I briefly thought i had a counterexample, on the grounds that

  "$\exists V_\omega$' is, for each $n$, equivalent to

  "The least fixpoint for $x \mapsto (\mathcal{P}_{\aleph_0})^n(x)$ exists"

  and that last assertion is stratifiable-mod-$n$. So it ought to be invariant, but it isn't, beco's of Holmes' clever permutation.

  However, the least fixpoint for $x \mapsto (\mathcal{P}_{\aleph_0})^n(x)$ isn't $V_\omega$. It's the set of sets of rank a multiple of $n$.. Duh.

- Randall has just (4/vi/2016) pointed out to me that $\mathrm{TC}_n\mathrm{T}$ is in some sense the same theory as NFU + $|V| = |\mathcal{P}^n(V)|$. It could be a good idea to spell this out. Evidently any model of NFU + $|V| = |\mathcal{P}^n(V)|$ will give rise to a model of $\mathrm{TC}_n\mathrm{T}$. The other direction looks a lot more complicated.

- In a model of $\mathrm{TC}_k\mathrm{T}$ one can sensibly ask, for any $m$, whether or not Ambiguity holds for formulæ that are stratifiable-mod-$k \cdot m$.

- André Pétry suggests a generalisation of a result of his-and-mine alluded to earlier ([9], [17], and [18]) to the effect that if two structures are elementarily equivalent for formulæ that are stratifiable-mod-$n$ then they have stratimorphic (as it were) ultrapowers.

- One could investigate whether the construction of [11] could be modified to encompass expressions that are stratifiable-mod-$n$. That looks messy.

- There are natural settings where one encounters embeddings that are elementary for stratifiable formulæ, and where one might hope to get embeddings that are elementary for some of these larger classes of formulæ. CO models is one setting: the embedding from the ground model into the hereditary low sets is elementary for stratifiable formulæ. (That particular example is probably not a good one, because if the inclusion embedding is elementary for formulæ that are stratifiable-mod-$n$ for even one $n$ then the hereditarily low sets cannot contain any Quine atoms). For another, let $\mathfrak{M}$ be a structure for $\mathcal{L}$. Consider the class of those $m \in M$ s.t. $m$ is fixed by all permutations of $M$ that, for all $n$, are $j^n$ of something. It's an elementary substructure as long as it's extensional. Now use instead those permutations $\pi$ of $M$ s.t. $j^m \pi = \mathbb{1}$. Now the class of fixed things is a substructure elementary for expressions that are stratifiable mod $m$ (again, assuming extensionality).

• Str(ZF) is the theory axiomatised by the stratifiable axioms of ZF; by analogy $\mathrm{str}_n(\mathrm{ZF})$ will be the theory axiomatised by those axioms of ZF that are stratifiable-mod-$n$. ZF can be interpreted in str(ZF) + IO. (IO is the axiom "every set is the same size as a set of singletons"). Observe that IO is a theorem of $\mathrm{str}_n(\mathrm{ZF})$, since it proves that $\iota^n \upharpoonright x$ exists for all $x$, so every set is the same size as a set of singletons. Indeed even $\mathrm{str}_n(\mathrm{Z})$ (the stratifiable-mod-$n$ fragment of Zermelo set theory) proves IO. So ZF can be interpreted in $\mathrm{str}_n(\mathrm{ZF})$. At this stage we cannot see how to prove that $\mathrm{str}_n(\mathrm{ZF}) = \mathrm{ZF}$. There are parallel questions about the fragments of Mac.

• Stratified parameter-free $\Delta_0$ $\in$-induction seems to prove no more than the nonexistence of a universal set. How about stratifiable-mod-$n$ parameter-free $\in$-induction... what does that do? One might hope that it would prove the nonexistence of $\in$-loops of circumference $n$ but we can't see it offhand. But in any case we should start with the case $n = 2$ in order to not drown *immediately* in the deep end. We noted in section 3 that the collections `I` and `II` as in [13] are both the extensions of expressions that are stratifiable-mod-2. So stratifiable-mod-$n$ parameter-free $\in$-induction will imply $\in$-determinacy. (though that induction is not $\Delta_0$...) Needs looking into.

Stratifiable parameter-free $\in$-induction implies the nonexistence of the universal set. (If none of your members are the universal set, you can't be either). It's not known if the converse holds. However the strengthening of the converse one would consider in this context, namely "the non-existence of the universal set implies $\in$-induction for parameter-free formulæ that are stratifiable-mod-$n$" clearly does not go through: $\in$-induction for parameter-free formulæ that are stratifiable-mod-$n$" implies $(\forall x)(x \not\in^2 x)$, and that clearly doesn't follow from the nonexistence of $V$.

• Suppose we add to our favourite theory of wellfounded sets a scheme of $\in$-induction for formulæ that are stratifiable-mod-$n$, for some or all $n$. Is it the case that any such model is first-order indistinguishable from a wellfounded model? Can we prove anything with that flavour ...? A: by proposition 2 we could prove that every set is wellfounded.

• Every weakly stratifiable theorem of first-order logic has a cut-free weakly stratifiable proof; every stratifiable theorem of first-order logic has a stratifiable proof (Crabbé, [4]); are there analogues for stratification-mod-$n$? Every theorem of first-order logic that is stratifiable-mod-$n$ has a proof that is stratifiable-mod-$n$? Crabbé thinks so. Why should it *not* work, after all?

On the other hand we should not expect a stratifiable-mod-$n$ analogue of Crabbé's result that SF is consistent.

• There is an old question about whether the atoms of a model of NFU can be indiscernible. We know that they are indiscernible wrt stratifiable formulæ;

now that we've started looking into stratification-mod-$n$ it is natural to wonder whether one might be able to show that the atoms of a model of NFU must be indiscernible wrt expressions that are stratifiable-mod-2. At this stage it's not looking hopeful.

• We should investigate the consistency results relative to T$\mathbb{Z}$T obtained by omitting types, to see how many of them work for TC$_n$T. They make heavy use of Coret's lemma. Coret's lemma tells us how permutations preserve stratifiable formulæ. Any old permutation works. In the NF context we know that if we want to preserve all formulæ then we can't use any-old-permutation but only $\in$-automorphisms. Working in TC$_n$T we want to preserve formulæ that are stratifiable-mod-$n$, and that means using permutations $\pi$ s.t. $\pi = j^n(\pi)$, and such permutations are not just lying around. TC$_n$T really does behave more like NF than like T$\mathbb{Z}$T.

• There is the old question of whether or not Amb$^n$ is equiconsistent with NF. Suppose we work in KF, and consider TC$_2$T to keep things simple initially. Suppose we have an $x$ with $|x| = |\mathcal{P}^2 x|$. Is that going to give us a model of NF? Let $\alpha$ be the cardinal of such an $x$. Can we prove that $\alpha = 2^{T\alpha}$? We suspect not, because that would probably say something about theorems in TC$_2$T. A useful thought is the fact that $\alpha$ is $\beth_n$ of something for all concrete $n$. So we certainly have $\alpha = \alpha+1$, $\alpha = \alpha+\alpha$, $\alpha = \alpha\cdot\alpha$. The plan is to use these equations to show that $x \sqcup \mathcal{P}(x)$ gives us a model of NF. So we want $T(2^{\alpha+2^{T\alpha}}) = \alpha + 2^{T\alpha}$. Now $T(2^{\alpha+2^{T\alpha}}) = 2^{T\alpha} \cdot 2^{2^{T^2\alpha}}$.

So we want $\alpha + 2^{T\alpha} = 2^{T\alpha}\cdot\alpha$, and we hope to get it from the good behaviour of $\alpha$. We have $\alpha = \alpha^2$ so we get $2^{T\alpha} = 2^{T\alpha^2} = (2^{T\alpha})^{T\alpha}$ which looks hopeful but isn't exactly what we want. The warning sign is that if this worked it would show that $2^{T\alpha}$ absorbs $\alpha$ and that sounds extremely implausible.

But even if $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$ it wouldn't help. We can exploit Bernstein's Lemma to show that we would have $\alpha = 2^{T\alpha}$ or – at the very least – that each $\leq^*$ the other, which is just as bad, as follows.

If we have $\alpha + 2^{T\alpha} = 2^{T\alpha}\cdot\alpha$ then Bernstein's Lemma gives $\alpha \leq 2^{T\alpha} \vee 2^{T\alpha} \leq^* \alpha$ and $\alpha \leq^* 2^{T\alpha} \vee 2^{T\alpha} \leq \alpha$ so a case analysis gives $\alpha = 2^{T\alpha} \vee \alpha \leq^* 2^{T\alpha} \leq^* \alpha$ which gives $2^\alpha = 2^{2^{T\alpha}} = T\alpha$, which is altogether too strong.

One has the impression that KF really does not want to prove that if there is $x$ with $|x| = |\mathcal{P}^n(x)|$ then there is an $x$ with $|x| = |\mathcal{P}(x)|$. The moral of this seems to be that TC$_2$T is not as much like NF as it might be.

• Consider "$\Box$(Duality for sentences that are stratifiable-mod-2)"

Is this consistent? Does it imply AC$_2$?

• ZF + Foundation and ZF + antifoundation are alike extensions of ZF + Coret's axiom "every set is the same size as a wellfounded set" conservative for stratifiable sentences. (See [14]). Does this hold also for sentences that are stratifiable-mod-$n$?

Checking this last one should be simple!

### 10.1 The full symmetric group on the Universe

The idea is that this section will become a self-contained paper with the above title.

Things to cover:

Nathan's partial order; universal involutions, universal permutations of all orders

Pull in stuff from chapter about permutation models in NFnotesredux.tex

Bowler-Forster on normal subgroups

First-order theories of infinite symmetric groups

theorem on permutation models and normal generating subsets.

Existence of polarities implies AC for wellordered sets of pairs.

Existence of universal permutations needs $V$ to be $V$ (or at least a power set) beco's of the appearance of $j$ in the definition of some of the universal permutations, not merely an infinite set.

Pull in the stuff about centralisers of $J_n$

Also pull in the stuff from NFnotesredux.tex about how there can be finite non-singleton orbits under $J_n$ if AxInf fails. Look for 'two-orbit'.

We can define the partial order for any symmetric group but in an NF context it takes on some interesting structure. CUS is synonymous with ZF but NF is not. This is something to do with the fact that NF has much more to say about Big sets than CUS does; for example, according to NF (but not CUS) the symmetric group on the universe is a set and there is quite a lot one can say about it. These things are almost certainly not going to recounted by ZF and so we'd better prick up our ears and pay attention.

## References


[1] Nathan Bowler and Thomas Forster "Internal Automorphisms and Antimorphisms of Models of NF" Journal of Symbolic Logic *to appear*

[2] Nathan Bowler and Thomas Forster, "Normal Subgroups of Infinite Symmetric Groups, with an Application to Stratified Set Theory". Journal of Symbolic Logic **74** (march 2009) pp 17–26.

[3] Crabbé, M. "Typical ambiguity and the Axiom of Choice". Journal of Symbolic Logic **49**, (1984) pp. 1074–1078.

[4] Crabbé, M. "Stratification and Cut-elimination" Journal of Symbolic Logic **56**, (1991) pp. 213–226.

[5] Olivier Esser and Thomas Forster, "Relaxing stratification" *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), pp. 247–258. Also available from `www.dpmms.cam.ac.uk/~tef10/relaxing.pdf`

[6] Olivier Esser and Thomas Forster, “Seeking Structure for the Collection of Rieger-Bernays Permutation models” unpublished but available from `www.dpmms.cam.ac.uk/~tef10/permstalk.pdf`

[7] Thomas Forster “The Burali-Forti Paradox” The Reasoner **17** 5, sept 2023 pp 40–41.

[8] Thomas Forster “Set Theory with a Universal Set” Oxford Logic guides **20**. Oxford University Press 1992.

[9] Thomas Forster, “Permutation Models and Stratified Formulæ, a Preservation Theorem” Zeitschrift für Mathematische Logic und Grundlagen der Mathematik, **36** (1990) pp 385–388. Also available from `www.dpmms.cam.ac.uk/~tef10/strZF.pdf`

[10] Assorted stuff on CO models.

[11] Thomas Forster, “AC fails in the natural analogues of $L$ and the cumulative hierarchy that model the stratified fragment of ZF”. *Contemporary Mathematics* **36** 2004. Also available from `www.dpmms.cam.ac.uk/~tef10/zmlmany.pdf`

[12] Thomas Forster “A new Datatype of `scansets` and some Applications: Interpreting Mac in KF” unpublished but available at `www.dpmms.cam.ac.uk/~tef10/scansets.pdf`

[13] Thomas Forster “Games Played on an Illfounded Membership Relation”. in *A Tribute to Maurice Boffa* ed Crabbé, Point, and Michaux. Supplement to the December 2001 number of the *Bulletin of the Belgian Mathematical Society*.

[14] Thomas Forster ZF + “Every Set is the same size as a Wellfounded Set” *Journal of Symbolic Logic* **58** (2003) pp 1–4.

[15] M. Randall Holmes “The consistency of NF”

[16] M Randall Holmes “The Equivalence of NF-style Set theories with “Tangled” type theories; the construction of $\omega$-models of predicative NF (and more)”. Journal of Symbolic Logic **60**, pp. 178–189.

[17] Pétry, A. “Une charactérisation algébrique des structures satisfaisant les mêmes sentences stratifiées” Cahiers du Centre de Logique (Louvain-la-neuve) **4**, (1982) pp. 7–16.

[18] Pétry, A. “Stratified languages” Journal of Symbolic Logic **57**, (1992) pp. 1366–1376.

[19] Zuhair Al-Johar, M. Randall Holmes and Nathan Bowler “The Axiom Scheme of Acyclic Comprehension” Notre Dame J. Formal Logic `http://projecteuclid.org/info/euclid.ndjfl` **55**, Number 1, (2014), 1–155. `http://projecteuclid.org/euclid.ndjfl/1390246432`.