A Useful(?) Reformulation of Paris-Harrington

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ABSTRACT

A formulation of Paris-Harrington ([2]) is described which is stratified and differs from finite Ramsey purely in the quantifier prefix.

Various people[1] have commented that the concept of a *relatively large set of natural numbers* is unstratified, and in [1] I mused about whether or not the extra strength of P-H over finite Ramsey was to do with this failure of stratification. In the present—self-contained—sequel I shall show that it is not: P-H has a stratified formulation, and one that presents it as an analogue of finite Ramsey in a natural and novel way.

I was able to exhibit in [1] formulations of P-H and Finite Ramsey which differed only in their quantifier prefix, as below, quoted from [1]:

Finite Ramsey:

> *For all n, m, j in $\mathbb{N}$*
> *There is k in $\mathbb{N}$ so large that*
> *For every set X of size k and*
> *For every m-colouring χ of $[X]^j$*
> *there is an enumeration e of X and*
> *there is $X' \subseteq X$ with $|X'| = n$ and X' monochromatic with respect to χ and relatively large with respect to e.*

Paris-Harrington:

> *For all n, m, j in $\mathbb{N}$*
> *There is k in $\mathbb{N}$ so large that*
> *For every set X of size k and*
> *For every m-colouring χ of $[X]^j$ and*
> *For every enumeration e of X*
> *there is $X' \subseteq X$ with $|X'| = n$ and X' monochromatic with respect to χ and relatively large with respect to e.*

[1] `Subject: [FOM] PA Incompleteness; Sun, 14 Oct 2007 10:02:58 -0400`: Harvey Friedman wrote
"In "relatively large", an integer is used both as an element of a finite set and as a cardinality (of that same set). This is sufficiently unlike standard mathematics, that an effort began, at least implicitly, to find PA incompleteness that did not employ this feature, or this kind of feature."

The only difference is in the quantifiers in the fifth line. Locating the difference in the quantifier prefix suggests that stratification is not the key to understanding the situation. However, both these formulations make use of the concept of *relatively large subset*, and so are not stratified.

This unstratification is a bother. To NFistes it's worse than a bother, it looks like a mistake. Every NFiste "knows" that Arithmetic is stratified. So an NFiste looking at P-H is always going to expect to see a stratified reformulation of it.

I can now exhibit a pair of *stratified* formulations of Finite Ramsey and P-H. This confirms that stratification plays no role in the extra strength of P-H. However, more striking (more *mainstream* perhaps) is the observation that these two theorems differ only in their quantifier prefix.

First, some notation: $\text{Ƅ}(x)$ is $\{y : y \cap x \neq \emptyset\}$. '$\text{Ƅ}$' is an upside-down '$\mathcal{P}$' reflecting the fact that the operation is dual to power set: $\text{Ƅ}(x) = \overline{\mathcal{P}(\overline{x})}$. This makes it clear that $\text{Ƅ}(x)$ is going to be a proper class according to ZF, and this fact frees us to abuse notation by writing '$\text{Ƅ}(x)$' for $\{y \subseteq X : y \cap x \neq \emptyset\}$ when $x \subseteq X$, as long as X is clear from context.

The difference between Ramsey and P-H is usually taken to be that Ramsey says there is a monochromatic set, and that P-H says that there is a monochromatic set with special properties. The new thought is that we should think of P-H as saying not so much that *there is a monochromatic set with special properties,* but rather that *there are* **lots** *of monochromatic sets*: the collection of monochromatic subsets of X is *large* in the sense that, for every total order $<$ of X, it meets Ƅ of the initial segment containing the first n elements of $\langle X, < \rangle$. That sounds like a quantifier.

Finite Ramsey says

> *For all* $n, m, j \in \mathbb{N}$
> *There is* $k \in \mathbb{N}$ *so large that*
> *For every set* X *of size* k
> *For every* m*-colouring* χ *of* $[X]^j$
> *There is* $X' \subset X$ *with* X' *monochromatic with respect to* χ.

We can rephrase the last line to get

> *For all* $n, m, j \in \mathbb{N}$
> *There is* $k \in \mathbb{N}$ *so large that*
> *For every set* X *of size* k *and*
> *For every* m*-colouring* χ *of* $[X]^j$
> *The set* M_n^χ *of* n*-sized subsets of* X *monochromatic for* χ *is nonempty.*

Now for the new formulation of P-H:

> *For all* $n, m, j \in \mathbb{N}$
> *There is* $k \in \mathbb{N}$ *so large that*
> *For every set* X *of size* k *and*
> *For every* m*-colouring* χ *of* $[X]^j$

The set M_n^χ of n-sized subsets of X monochromatic for χ meets everything in $Ь``[X]^n$.

...and the difference between these two [purely in the fifth line] is that one says that the set of monochromatic subsets of size n is nonempty, whereas the other says that it meets every member of a fairly large set.

It's probably worth saying a few words about why this version of P-H is equivalent to the usual version that asserts the relative largeness of the monochromatic set. A subset of $\mathbb{N}$ of size n is relatively large simply if it has a member smaller than n. But this is simply to say that an n-sized subset $X' \subseteq X$ is relatively large with respect to an ordering $<$ iff one of its members is among the first n members of X according to $<$. For the other direction, if Y meets some element $X' \in [X]^n$ then it is relatively large with respect to any enumeration of X that counts X' using only the naturals $\leq n$.

This formulation prompts some natural questions. Does this version have a slick compactness proof? Does it give a slick proof of Con(PA)?

References

[1] Thomas Forster "Paris-Harrington in an NF context", in OHYAST (100 Years of Axiomatic Set Theory) *Cahiers du Centre de Logique* 2010 **17** pp 97–109. Also available from `www.dpmms.cam.ac.uk/~tef10/parisharringtontalk.pdf`

[2] Paris, J. and Harrington, L. A Mathematical Incompleteness in Peano Arithmetic. In Handbook for Mathematical Logic (Ed. J. Barwise). Amsterdam, Netherlands: North-Holland, 1977.