Five grades of Typical Ambiguity

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Contents

The purpose of this brief note[1] is to reprise Specker's note [5] on Typical Ambiguity, and to discuss what might happen to those ideas in a constructive setting. The motivation for this twist is that there are obvious realizers for the ambiguity axioms, so there *might* be an argument for the consistency of the constructive fragment *i*NF of NF that is not at the same time an argument for the consistency of NF. We now finally know – thanks to Holmes [4] – that NF is consistent, but it would be very pleasing if there should turn out to be two routes to this result not just the one.

The reader may be aware ([2] and [3]) that if a contradiction can be derived from a set A of axioms then one can derive a contradiction from $\{\neg\neg\phi : \phi \in A\}$. Why does this not promptly scupper the possibility of a classical theory that refutes ambiguity having a constructive fragment that is consistent with ambiguity? The point is that Glivenko's result does not hold for predicate logic. This chink leaves open the possibility that there might be classical theories that refute ambiguity but whose constructive fragments are consistent with it, In this note i will show at least that one of the example theories in [5] is such a theory:

> **There is a classical theory that contradicts Typical Ambiguity but whose constructive fragment is consistent with Typical Ambiguity.**

The most general setting for ideas like these is a first-order language $\mathcal{L}$ with a bijection $\sigma : \mathcal{L} \longleftrightarrow \mathcal{L}$ that commutes with quantifiers and connectives, and an $\mathcal{L}$-theory T such that σ is an automorphism of T in the sense that, for all $\phi \in \mathcal{L}$, $T \vdash \phi$ iff $T \vdash \sigma(\phi)$. (Henceforth we will write 'ψ^σ' rather than '$\sigma(\psi)$'). The two Specker articles are essential reading. T$\mathbb{Z}$T (which is the strongly typed set theory with levels

[1] Thanks to Randall Holmes and Stephen Mackereth for helpful comments

indexed by $\mathbb{Z}$, the theory in [7][2]) is such a theory. Indeed T**Z**T is likely to be our main preoccupation in what follows, and the automorphism of $\mathcal{L}$(T**Z**T) that is of interest to us is the operation that bumps up the type of a formula by one. Traditionally we write this automorphism with a '+' symbol: thus ϕ^+ is the result of lifting all type subscripts in ϕ by one.

1 The Five Grades

The phrase 'typical ambiguity' is often used in contexts like this, and it comes in (at least!) five grades.

Grade (i)	$T \vdash \phi$ iff $T \vdash \phi^+$;
Grade (ii)	$T \cup \{\phi \longleftrightarrow \phi^+\}$ is consistent for each ϕ;
Grade (iii)	$T \cup \sum_{\phi \in \mathcal{L}} \phi \longleftrightarrow \phi^+$ is consistent;
Grade (iv)	$T \vdash \sum_{\phi \in \mathcal{L}} \phi \longleftrightarrow \phi^+$;
Grade (v)	T has an ambiguous model.

Grade (v) is complicated to state in general but in the case of interest here (T**Z**T) an ambiguous model (*glissant* in French) is a model with an automorphism that sends elements of type (level) i to elements of level $i + 1$. Any such model of T**Z**T will give a model of NF as per Specker [6].

> It may be worth noting *en passant* that – as Specker points out – if the + operation is an involution[3] then (ii) implies (iii). However the + operation in play in TZT is of infinite order so this observation is not very useful to us, tho' it is worth keeping in mind for later use in a more general context.

Classically there is a theorem of Specker's [5] that says that grade (iii) implies grade (v)[4]. (A theory of grade (iii) can be extended to a theory of grade (iv), and grade (iv) implies grade (v) by general model-theoretic nonsense, and we supply no proof). This gives a reduction of Con(NF) to the assertion that T**Z**T is grade (iii)). Altho' Marcel Crabbé showed in [1] that T**Z**T is grade (ii), unfortunately there is no obvious reason to believe that it is grade (iii) (tho' in fact it is) and it manifestly isn't grade (iv). Classically you can have theories that are grade (ii) but have no extensions that are grade (iv). Specker [5] supplies examples which we will consider below.

[2]In [7] this theory is called the *Theory of Negative Types* and for many years was called 'TNT'. Nowadays the notation 'T**Z**T' is preferred, leaving 'TNT' to denote the analogous theory with types indexed by the negative integers. . . which would not otherwise have a name. One wants to distinguish the two because it is far from clear that every model of TNT can be "extended upwards" to a model of T**Z**T.

[3]There is probably something similar one can say if + is an operation of finite order; should say it!

[4]Specker does not use this 'grade' terminology.

1.1 Specker's Example of a grade (ii) Theory that is not grade (v)

We next consider Specker's Example of a Theory that is grade (ii) but not grade (v), from: Specker [5].

The language has levels indexed by $\mathbb{Z}$ (so each variable is restricted to range over one level only), and it's a first order language with equality but no nonlogical vocabulary. Our theory T will have two axiom schemes:

(1) There are precisely 1, 2, or 3 elements of each level;
(2) There are not equally many elements of level k and of level $k + 1$.

To be formal about it let us write

'$\exists! x_i$' for '$(\exists x_i)(\forall y_i)(x_i = y_i)$',
'$\exists_2! x_i$' for '$(\exists x_i, y_i)(x_i \neq y_i \wedge (\forall z_i)(x_i = z_i \vee z_i = y_i))$' and
'$\exists_3 x_i$' similarly.

At each level i, T has axioms

- $(\forall a_i b_i c_i d_i)(a_i = b_i \ \vee\ a_i = c_i \ \vee\ a_i = d_i \ \vee\ b_i = c_i \ \vee\ b_i = d_i \ \vee\ c_i = d_i)$
- $\neg(\exists! x_i \wedge \exists! x_{i+1})$
- $\neg(\exists_2! x_i \wedge \exists_2! x_{i+1})$
- $\neg(\exists_3! x_i \wedge \exists_3! x_{i+1})$
- $(\exists x_i)(x_i = x_i)$

This T is a theory in (many-sorted) first-order logic, but many of its features can be captured in a propositional theory T_{prop} in a language $\mathcal{L}$ with propositional letters p_i, q_i and r_i for all $i \in \mathbb{Z}$. The theory T_{prop} has two schemes:

- $(p_i \wedge \neg q_i \wedge \neg r_i) \ \vee\ (\neg p_i \wedge q_i \wedge \neg r_i) \ \vee\ (\neg p_i \wedge \neg q_i \wedge r_i)$
- $\neg(p_i \wedge p_{i+1}) \ \wedge\ \neg(q_i \wedge q_{i+1}) \ \wedge\ \neg(r_i \wedge r_{i+1})$.

one instance of each for each $i \in \mathbb{Z}$. (Secretly p_i says that there is precisely one object at level i; q_i says that there are precisely two, and r_i says there are precisely three. This interprets T into T_{prop}.) We will show that T_{prop} (and therefore T) is grade (ii) but not grade (iv).

Recall that we write ψ^+ for the result of increasing every subscript in ψ by 1. We can also define this '+' operation to propositional valuations: if f is a valuation defined on the letters in $\mathcal{L}$ then f^+ is the valuation $v^+ \mapsto f(v)$. This ensures that f satisfies ψ iff f^+ satisfies ψ^+.

T_{prop} considered as the deductive closure of these axioms (a set of formulæ), has lots of automorphisms (one can permute the letters $\{p, q, r\}$) but the sole automorphism of interest to us is the one that sends every p-variable 'p_i' to 'p_{i+1}', and q- and r-letters similarly. This is the automorphism we have been writing with a '+' sign. Altho', for all ψ, we have: $T_{\text{prop}} \vdash \psi$ iff $T_{\text{prop}} \vdash \psi^+$, nevertheless we do not have $T_{\text{prop}} \vdash \psi \longleftrightarrow \psi^+$ for all ψ. Thus T is grade (i) but is not grade (iv). It remains to be shown that it is also grade (ii).

Remark 1
(a) $T_{\text{prop}} \cup$ *the scheme of biconditionals* $\psi \longleftrightarrow \psi^+$ *is inconsistent;*
(b) Each biconditional $\psi \longleftrightarrow \psi^+$ *is individually consistent with* T_{prop}.

Proof:

(a) is pretty obvious. This shows that a grade (i) theory need not be grade (iii)

(b) Given an arbitrary $\mathcal{L}$-formula ψ we will find a T_{prop}-valuation satisfying $\psi \longleftrightarrow \psi^+$. (A valuation is a function from the propositional letters in $\mathcal{L}$ to {`true`, `false`}; a T_{prop}-valuation is one that validates/accepts every theorem of T_{prop}).

Suppose (with a view to obtaining a contradiction) that every T_{prop}-valuation satisfies precisely one of ψ and ψ^+. Now T_{prop} says that precisely one of p_i, q_i and r_i will be true at level i, so we can think of a T_{prop}-valuation as *inter alia* a function that tells us, given i, which of p_i, q_i and r_i is true at level i.

Let f be the valuation that responds successively: $\ldots p\,q\,r\,p\,q\,r \ldots$ (with period 3) as the levels increase, and consider the three valuations f^+, f^{++} and $f^{+++} = f$. (Quick reality check: these are all T_{prop}-valuations). Recall that f sat ψ iff f^+ sat ψ^+ and so on. By the *reductio* assumption each of f, f^+ and f^{++} satisfies precisely one of $\{\psi, \psi^+\}$. Without loss of generality f satisfies ψ but not ψ^+; then f^+ satisfies ψ^+ but not ψ; f^{++} satisfies ψ but not ψ^+, and finally f^{+++} satisfies ψ^+ but not ψ. But $f^{+++} = f$.

> I think this is why we need three possible sizes, rather than two ... to make the parity trick work.

■

We observe without proof that the interpretation $T \hookrightarrow T_{\text{prop}}$ enables us to port remark 1 to (our formulation of) Specker's original first-order theory T.

> The above proof of remark 1 is not the proof on p. 8 of the English translation of [5], but rather a side-effect of my attempts to understand that proof.

2 The Constructive Setting

Thus, manifestly, classically neither grade (i) nor grade (ii) implies grade (iv). (All this is in in [5].) However if we are thinking constructively then the + operation on proofs gives obvious candidates for realizers of the biconditionals in (ii) and (iii) and thereby gives us reason to believe that *constructively* grade (i) *might* imply grade (iv); thus we are moved to go looking for a version of Specker's theorem for constructive logic.

In propositional logic any contradiction provable classically is provable constructively, so there is no hope for a constructive version of Specker's theorem for *propositional* logic. However, this doesn't tell us that there can be no version for constructive *first-order* logic. For example: it may be that the first-order theory T (which – remember – was Specker's original example) has a constructive version to which we can consistently add an axiom scheme of typical ambiguity. (In fact I cunningly chose the axiomatisation above with precisely this possibility in mind, so that we can use the same example.)

It turns out that this is indeed the case: if we use a constructive logic then this T remains consistent when we add the scheme $\phi \longleftrightarrow \phi^+$.

The constructive version *i*T of Specker's theory T will have the same axioms, or at least axioms that are classically equivalent to the axioms of T, embedded in a constructive logic. It will say, at each level, that there are one, two or three things; yet there is not precisely one thing, nor precisely two things, nor precisely three things.

(The theological vapours emanating from this trilemma should alert us to the dangers ahead).

> Reflect that there is no point in trying to set up a constructive version of T_{prop}. By Glivenko [2] and [3] any contradiction provable in a classical propositional theory is provable also in its constructive fragment.

We want axioms to say that, at each level

- there is at least one but not as many as four distinct things,
- there is not precisely one thing,
- nor precisely two,
- nor precisely three.

and that adjacent levels have different numbers of members.

So, first of all, we have an axiom that says there is a thing such that if we pick up four things then two of the four are equal. That ensures that there is at least one thing but not as many as four distinct things. We want that in the weakest possible form, co's it's going to be an axiom of the quotient theory, which we want to be consistent. Then we have the three expressions that say "there are precisely three things", "there are precisely two ...". When we add the ambiguity scheme we have to consider the instances of it that involve the assertions "there are precisely k things". If these biconditionals are to be true then – given that we require adjacent levels to have *different* numbers of inhabitants – all these assertions (that a given level has precisely k elements) all have to be *false*.

Therefore we want our formulation of these "there are precisely k elements" to be as *strong* as possible beco's they are going to have a *negative* occurrence in the quotient theory.

Our first axiom has to say

$$(\exists x)(\forall yzw)(x = y \ \lor \ x = z \ \lor \ x = w \ \lor \ y = z \ \lor \ y = w \ \lor \ z = w)$$

So we adopt the classically equivalent but constructively weaker

$$\exists x \forall yzw \neg(x \neq y \ \land \ x \neq z \ \land \ x \neq w \ \land \ y \neq z \ \land \ y \neq w \ \land \ z \neq w)$$

which has been made nearly as weak as possible. We could have further weakened the '$\exists$' to '$\neg\forall\neg$' but it turns out that that is not necessary.

Then we have the three assertions "there are precisely 1, 2, 3 elements" which have to be made as *strong* as possible. Let's use the following abbreviations to protect our sanity

$\texttt{notfour}_i$	$\exists x_i \forall y_i z_i w_i \neg(x_i \neq y_i \land x_i \neq z_i \land x_i \neq w_i \land y_i \neq z_i \land y_i \neq w_i \land z_i \neq w_i)$
$\texttt{notone}_i$	$\neg\exists x_i \forall y_i y_i = x_i$
$\texttt{nottwo}_i$	$\neg(\exists x_i y_i)(x_i \neq y_i \land (\forall z_i)(z_i = x_i \lor z_i = y_i))$
$\texttt{notthree}_i$	$\neg(\exists x_i y_i z_i)(x_i \neq y_i \land y_i \neq z_i \land x_i \neq z_i \land (\forall w_i)(w_i = x_i \lor w_i = y_i \lor w_i = z_i))$

And our version *iT* of Specker's theory *T* has the four axiom schemes

- $\neg(\neg\texttt{notone}_i \ \land \ \neg\texttt{notone}_{i+1})$;
- $\neg(\neg\texttt{nottwo}_i \ \land \ \neg\texttt{nottwo}_{i+1})$;
- $\neg(\neg\texttt{notthree}_i \ \land \ \neg\texttt{notthree}_{i+1})$; and
- $\texttt{notfour}_i$.

This theory is grade (ii) (it's a subset of a grade (ii) theory).

We now come to the crucial difference from the classical case. If we add the ambiguity scheme $\phi \longleftrightarrow \phi^+$ to *i*T the result is consistent. It proves $\neg\mathtt{notthree}_i$, $\neg\mathtt{nottwo}_i$ and $\neg\mathtt{notone}_i$ at each level i.

The consistency of this conjunction looks fairly daunting, but it's actually quite simple: let each level contain three things which are notnotequal.

We need a possible world model for this!

We can now consider the "quotient" theory, the one-sorted theory obtained from *i*T by simply erasing the type indices. It contains

$\mathtt{notfour}$: $\exists x \forall yzw \neg(x \neq y \wedge x \neq z \wedge x \neq w \wedge y \neq z \wedge y \neq w \wedge z \neq w)$
$\mathtt{notone}$: $\neg\exists x \forall y y = x$
$\mathtt{nottwo}$: $\neg(\exists xy)(x \neq y \wedge (\forall z)(z = x \vee z = y))$
$\mathtt{notthree}$: $\neg(\exists xyz)(x \neq y \wedge y \neq z \wedge x \neq z \wedge (\forall w)(w = x \vee w = y \vee w = z))$

This, too, is consistent. For example a model containing three things which are all notnotequal.

So we have an example of a classical theory that refutes typical ambiguity but whose constructive fragment is consistent with typical ambiguity.

This inspires hope in the following possibility:

CONJECTURE 1
Whenever T is a constructive theory in a first-order language $\mathcal{L}$ admitting an automorphism σ such that
(i) $T \vdash \psi$ iff $T \vdash \psi^\sigma$ for every $\psi \in \mathcal{L}$; and
(ii) each formula $\psi \longleftrightarrow \psi^\sigma$ is consistent relative to T,
then T has a model admitting an automorphism corresponding to σ, and a consistent one-sorted quotient.

The reason for interest in this situation is that there might be classical theories whose constructive fragments satisfy some form of ambiguity for reasons like those seen above. Specifically the thought is that T**Z**T might be such a theory. Granted, we now know that NF is consistent (so *a fortiori* the constructive fragment is consistent too) but it would be nice to have an entirely separate proof of the consistency of *i*NF– the constructive fragment.

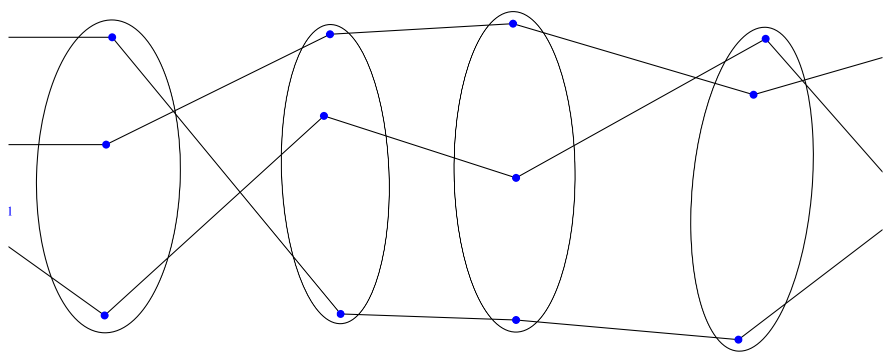
To clarify the situation and estimate the hopes for such a result we need to think a bit about what realizers are and what it is to raise the level of a formula.

2.1 Type-raising and Realizers

The thought that launched this discussion was the idea that realizers for formulæ in a strongly typed language such as $\mathcal{L}$(T**Z**T) were the kind of thing that type-raising operations could act on. If they are, then the raising of types certainly provides realizers for conditionals like $\phi \to \phi^+$.

We need to start by reflecting that there is no good notion of type-raising in propositional languages. And this is despite the artful way in which the reader was tricked on p. 3 into accepting T_{prop} as a typed propositional theory. Consider the (typical) axiom

$\neg(p_i \wedge p_{i+1})$. The trickery is the exploitation of the subscripts. As far as the propositional language is concerned the propositional letters have no internal structure: every permutation of the set of propositional letters of a propositional language is an automorphism of the language. One would expect that any permutation of the propositional letters would fix T_{prop} at least in the sense of sending it to something α-equivalent to it, but this is not so. If one thinks of T_{prop} as expressed in a language where none of the propositional letters have internal structure then T_{prop} in effect defines some structure on that language. It defines a three-place relation (the ellipses in the picture below) and a two-place relation (the left-to-right edges in the picture below). For a permutation of the set of propositional letters to fix T_{prop} (at least in the sense of sending it to something α-equivalent to it) then it must preserve the ellipses and the injective character of the edges.



But we don't need to make sense of type-raising in propositional logic, for reasons noted above. However we do need to think about precisely what we mean by "raising the types" in a *first-order* formula. Now a first-order theory is a set of *closed* formulæ so – *prima facie* – we only need to consider what it is to "raise the type" of a closed formula. Thus this + operation, if taken in the way explained above, is really an operation defined not on (closed) formulæ but on α-equivalence classes of (closed) formulæ. That is just as well, because the idea of raising types in an open formula is problematic. The simplest open formula is of course a naked variable. What happens, what does one get, if one raises the type of a variable? How do we decide which variable of the next level to replace a given variable with? If we are raising the type of a variable as part of an act of raising the type of a *closed* formula then it doesn't matter; the axioms are – all of them – closed formulæ, and all the results of performing this (somewhat nondeterministic) operation on ϕ will be α-equivalent to each other. If we are trying to raise the type of a naked variable it matters a great deal.

If type-raising is to be a realizer for $\phi \to \phi^+$ then realizers for ϕ have to be fairly syntactic objects – things that + can operate on. But + doesn't work very well on things with free variables. So it looks as tho' things with free variables can't have realizers. Or perhaps (and this would be a good outcome) things with free variables can have realizers all right, it's just that type-raising on them doesn't work.

> The usual way of thinking of variables is to take a variable to be – in the first instance – a letter from some standard alphabet (typically the Roman alphabet).

Such alphabets are always finite of course, so we augment them by use of a prime symbol: $'$, giving us variables such as x, x', y'' and so on.
In the context of typed set theory (TST, TZT…) these variables additionally are decorated with a subscript (or, in the early literature, a superscript) from $\mathbb{N}$ or $\mathbb{Z}$. If we think of variables that way then we can define the result of raising the type of a variable[5] v as the result of adding 1 to the numerical subscript in v. Then we define ϕ^+ to be the result of raising all the subscripts on all the variables in ϕ by precisely 1, with the result that + is an operation defined on formulæ themselves rather than merely on their α-equivalence classes.
There is a minor infelicity to this in that the presence of sort subscripts clogs up a place which has a long and honourable history of serving other useful purposes, such as notating sequences. This is notationally annoying but not mathematically substantial. We will not consider it further.
There is a second infelicity, of a different nature altogether. The attachment of subscripts provides extra information about the variables which – from the point of $\mathcal{L}$(TZT) – is spurious: the connection between 'x_2' and 'x_3' that is visible to us is not visible to the language.
It is this second infelicity that offers us an invitation to error. According to the new definition of the operation + it is defined on formulæ with free variables. On the face of it that ought to be harmless, so how is it an invitation to error?
As remarked above any TZT-proof of ϕ can be turned into a proof of ϕ^+. This means that "from $\vdash \phi$ infer $\vdash \phi^+$" is an admissible[6] rule of TZT. *However this is legitimate only if ϕ is a closed formula and there are no live assumptions in the proof.* If + is defined only on closed formulæ then the admissible rule is applicable to closed formulæ only. Once we extend + to formulæ with free variables then new formulæ come within the purview of the rule and it may cease to be admissible. This is in fact what happens.

References

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[3] Glivenko, V. "Sur quelques points de la Logique de M. Brouwer". Bulletin de L'Académie Royal de Belgique, série de la Classe des Sciences **15** 1929 pp 183–8.

[4] Randall Holmes: The consistency of NF

[5] Ernst Specker "Dualität" *Dialectica* **12** pp. 451–465. English translation available on `https://www.dpmms.cam.ac.uk/~tf/dualityquinevolume.pdf`

[6] Ernst Specker "Typical Ambiguity" in Logic, Methodology and Philosophy of Science *ed* Nagel *et al* Stanford 1960 pp 116–124.

[7] Wang, H. "Negative Types" `MIND` **61** pp. 366–8.

[5] Here 'v' is of course a variable ranging over variables(!)

[6] Wikipædia puts it very succinctly: "In logic, a rule of inference is admissible in a formal system if the set of theorems of the system does not change when that rule is added to the existing rules of the system. In other words, every formula that can be derived using that rule is already derivable without that rule, so, in a sense, it is redundant. The concept of an admissible rule was introduced by Paul Lorenzen (1955)."