Review of "Finsler Set Theory"

Paul Finsler was a geometer with an amateur interest in Foundations and Set Theory. Readers of this journal may have encountered one of the products of this interest when reading his celebrated anticipation of *Über Formal unentscheidbare Sätze ...* that appears in translation in Van Heijenoort or when reading the discussion of axioms of strong extensionality in the now classic monograph of Aczel [1988]. The appearance of a book containing English translations of all Finsler's published thoughts on this matter, together with commentaries by Ziegler and Booth gives us all a chance to see whether we have been missing anything by reading only the proto-Gödel article.

Finsler announces three desiderata for set theory, which he calls *axioms*. The first says (in translation) that it is always "uniquely determined" whether or not a given set is a member of another given set. Since Finsler does not seem to be concerned to make points against constructivism in mathematics (there is no evidence in this volume that he has any opinions on the matter) he presumably is not asserting excluded middle for $\in$. Nor does he appear to be insisting that a sensible axiomatic set theory should decide all expressions of the form $s \in t$ where s and t are terms, and perhaps one should regard this desideratum as expressing a view of the sort of things sets are, which nowadays does not need to be spelled out.

The third desideratum is that the universe of sets should be the largest thing satisfying the other two desiderata. To modern readers this is the most obscure of his *aperçus* and it is unlikely to be pursued by anyone primarily interested in learning more about set theory, though it may be of interest to historians of the subject. Finsler does not share the twentieth century taste for the axiomatic style and his material is obscure to those who do. His discussion of this desideratum takes as its point of departure the old saw that in mathematics existence is freedom from contradiction, but takes us nowhere.

It is the second desideratum that appeals most to contemporary tastes. It is an insistence that "isomorphic" sets be identical. Some call it an antifoundation axiom, but the expression "pseudofoundation axiom" would be better, since it is a *consequence* of foundation rather than something inconsistent with it. Better still would be "axiom of strong extensionality". This idea has been rediscovered several times (indeed it may go back earlier than Finsler) and has been treated in more detail by later authors. Axiomatising this insight is not as trivial as it looks. The difficulty is well illustrated by the mess the editors make of their discussion of the case of the two sets $a = \{a\}$; $b = \{a, b\}$. These two sets obviously should be identical according to Finsler's criterion, but it seems there can be no bijection between the transitive closures of a and b since the second has two members but the first has only one.

Dissolution of this puzzle necessitates first acknowledging that the notion of a bijection between $TC(a)$ and $TC(b)$ involves the equality relation between members of those two sets, and if a and b are illfounded (as here) there is no recursive procedure to resolve it. Next we define a **contraction** to be a fixed point for the following operation on equivalence relations. $X \sim^{+} Y$ iff $(\forall x \in X)(\exists y \in Y)(x \sim y) \wedge (\forall y \in Y)(\exists x \in X)(x \sim y)$. There are now various ways of formalising this insight.

(i) One is to say that the appropriate notion of a bijection between $TC(x)$ and $TC(y)$ is a relation $R \subseteq TC(x) \times TC(y)$ whose domain is $TC(x)$ and whose range is $TC(y)$ such that R^2 is a contraction on $\langle TC(x), \in\rangle$ and R^{-2} is a contraction on $\langle TC(y), \in\rangle$. There are two games $G_{x=y}$ that characterise this. In the first **equal** plays a relation $R \subseteq x \times y$ whose domain is x and whose range is y and **unequal** picks an ordered pair $\langle x_1, y_1\rangle \in R$ and they play $G_{x_1=y_1}$. If **unequal** is unable to play she loses; if **equal** is confronted by $\langle a, b\rangle$ where one component is empty and the other isn't then he loses. An apparently different but actually equivalent game was recently found by Isaac Malitz. In Malitz's game, **unequal** and **equal** play the game $G_{x=y}$ as follows. **unequal** picks $x_1 \in x$ (or $y_1 \in y$) and **equal** responds by picking $y_1 \in y$ (or $x_1 \in x$ if **unequal** had picked $y_1 \in y$). Then they play $G_{x_1=y_1}$. The first player who is unable to move loses.

(ii) Clearly one way of describing the difficulties raised by $a = \{a\}$; $b = \{a, b\}$ is to point out that that the equivalence relation that relates a to $\{a\}$ and to $\{a, b\}$ is a contraction. One can then enforce Finsler's criterion by an axiom that says that no transitive set admits a nontrivial contraction.

(iii) We can elaborate notions of n-similarity which generalise equinumerosity. Two sources for this are Tarski [1986] and Church [1974]. In Church's development two sets are 1-similar if they are the same size. Thereafter they are $(n+1)$-similar if there is a bijection betwen them such that identified elements are n-similar. We obtain an axiom with a Finsler flavour by insisting that for every set there is an n such that its n-equivalence class is a singleton, or (more plausibly perhaps) that for every set, the intersection of its n-equivalence classes (over all $n \in \mathbb{N}$) is a singleton. Tarski's idea is perhaps more elegant. Any permutation of a set acts on the power set of that set in the obvious way, by translation. Rather than think of this as two distinct actions of the one permutation it is better to think of this as two different permutations acting in the same (usual) way, so we can think of an operation, written j so that if π is a permutation of X, $j`\pi$ is a permutation of $\mathcal{P}`X$. No set is fixed by all π of course, but the empty set and the universe are both fixed by $j`\pi$ for any permutation π. We could say the empty set and the universe are 1-*symmetric*. Similarly Frege-Russell cardinals (equivalence classes under equipollence) are fixed by every permutation that is j^2 of something. We could say that Frege-Russell cardinals are 2-*symmetric*. Similarly we find more and more sets that are n-symmetric as n gets larger. Tarski was interested in these ideas as a way of applying the *Erlanger Programm* to Logic, but it could be used to provide an exegesis of Finsler's extensionality axiom. Just as with the Church definition of n-similarity we could consider an axiom asserting that for every set the only thing that is n-equivalent to it for every n is itself.

Like anyone who comes to the paradoxes of set theory, Finsler realised that vicious circles have at least *something* to do with them, and writes much about what he calls *circle-free sets*. His discussion of these topics is not helpful, because he never distinguishes between an interest in non-circular definitions of sets and an interest in sets which do not belong to their transitive closures. Nevertheless several charitable readers (the editors and Holmes [1996] among them) have felt that his writings on this anticipated Ackermann's set theory of [1956]. Finsler's one allusion to this is brief and ungracious. (The page reference to it in the index is also wrong, of which more later).

Finsler's views do not seem to evolve with developments in set theory. Indeed he seems completely unaware of what was happening around him, and he and his editors needlessly elaborate an in-house jargon for notions which already had perfectly adequate notations understood by other mortals ('$y \ \beta \ x$' for '$x \in y$'; "totally finite" for "hereditarily finite" and so on). Quine's idea of stratification is obviously related to Finsler's pretheoretic notion of a "circle-free set" and yet Quine is never alluded to by Finsler. This is not in itself so strange; after all, how many of us can honestly say we read all the primary literature we should, but it is hard to believe that Ernst Specker, a Zürich colleague of Finsler who did important work on Quine's set theory and who debated various minutiæ with Finsler at length (all this is in the book), did not expose him to these ideas. Even stranger is the fact that the editors never make the connection either.

There is a final, engagingly quixotic, chapter entitled "The combinatorics of non-wellfounded sets". (See Math. Rev. **37**, 6189) which readers who have to teach generating functions might wish to pillage for exercises-with-a-difference.

This book contains quite a lot of material of interest to those strongly drawn to set theory without foundation, but ultimately it is not really not a book one can recommend. Firstly, there are numerous typographical errors, most of them are trivial but some causing real confusion. Secondly the index is so inaccurate as to be of little use. There is an entry for Hans Lipps, for example, but no references to him on either of the indicated pages, and Poincaré appears not on page 22 but on page 41. I have been unable to determine any consistent pattern to the errors in the indexing, but readers who are determined to persist with this book could try, for a first approximation, adding 4 or 5 to all page references, particularly those in the latter part of the book. Again, this reviewer generally found the editors' contributions hagiographic and unilluminating. Finally, two major classes of reader are unlikely to want this book, namely historians and working set theorists. Historians will prefer to read the material in its original German (these originals can be found by consulting the web page below), and working set theorists might feel (as did this reviewer in the end) that even on the topics on which Finsler had something to say there is more interesting and profitable reading to be had elsewhere. Certainly the Lake and Reinhardt articles should be consulted by those who wish to pursue Ackermann set theory, as should the Ph.D. theses of those two scholars and Kirmayer if available; material concerning the axioms of strong extensionality foreshadowed by Finsler can be found by following pointers starting from Holmes' home page and of course in Aczel's excellent [1988]; readers in need of a second opinion or more detailed discussion can find a more extended and generally more sympathetic review of this book (Holmes, like Finsler, is a Platonist) in Holmes [1996].

This book raises questions. What was a geometer with an amateur interest in Set Theory doing discovering comets? Why is the study of the *violon d'Ingres* of a German mathematician working in Zürich supported by the anthroposophists of Dornach? Did he keep his telescope in one of their gingerbread houses? Irrelevant perhaps, but one would like to have answers.

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