Finite-to-one maps

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ABSTRACT

It is shown in ZF (without choice) that if there is a finite-to-one map $\mathcal{P}(X) \twoheadrightarrow X$, then X is finite.

In ZFC we can argue as follows: if there is a finite-to-one map from Y onto X, and X is infinite, then $|Y| \leq |X| \cdot \aleph_0 = |X|$, and $|Y| \geq |X|$ by AC, so $|Y| = |X|$. So there can be a finite-to-one map from $\mathcal{P}(X) \twoheadrightarrow X$ only if X is finite. (Here finite means "has cardinal in ω", and "finite-to-one" means that the preimage of every singleton is finite.) It is the purpose of this (self-contained) note to show that the result can be proved in ZF even without AC. The proof provided here uses replacement and cannot be conducted in Zermelo set theory. Nor does the proof generalise to show, for an arbitrary strongly inaccessible aleph κ that if there is a surjection $f : \mathcal{P}(X) \twoheadrightarrow X$ where $|f^{-1}``\{x\}| < \kappa$ for all $x \in X$ then $|X| < \kappa$.

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THEOREM: If there is a finite-to-one map $\mathcal{P}(X) \twoheadrightarrow X$, then X is finite.

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Proof: We prepare for the main proof with the following taster: if there is a finite-to-one map $\mathcal{P}(X) \twoheadrightarrow X$ and X is infinite then it must have a countable subset. Suppose X is an infinite set and $f : \mathcal{P}(X) \twoheadrightarrow X$ is finite-to-one. For $n \in \omega$ set $[X]^n = \{Y \subseteq X : |Y| = n\}$. For each n, there can be only finitely many m such that $f``[X]^n = f``[X]^m$, for if n were a counterexample then each $x \in f``[X]^n$ is $f$ of some unordered $m$-tuple for infinitely many $m$, contradicting the fact that $f$ is finite-to-one. Therefore infinitely many $f``[X]^n$ are distinct. So $\{f``[X]^n : n \in \omega\}$ is a countable subset of $\mathcal{P}(X)$ and $f``\{f``[X]^n : n \in \omega\}$ is the image of a countable set in a finite-to-one map and is therefore a countable subset of X.

The idea is to show that if $|X| \notin \omega$ then we can derive a contradiction by considering $\aleph^*(|X|)$, the least aleph not the size of a wellordered partition of X. We will exhibit a transfinite family of surjections $g_\alpha : X \twoheadrightarrow \omega_\alpha$, for all ordinals α. (We identify alephs with initial ordinals, and ordinals with the set of their predecessors, as is customary). To do this we will need three constructions, all of them seen in embryo form in the above proof that X must have a countably infinite subset. We fix X and a finite-to-one map $f : \mathcal{P}(X) \twoheadrightarrow X$ once for all.

1. Our first construction creates a surjection $X \twoheadrightarrow \omega_\alpha$ from a surjection $h : \mathcal{P}(X) \twoheadrightarrow \omega_\alpha$.

   Consider the map that sends each $x \in X$ to $h``(f^{-1}``\{x\})$. This sends each $x \in X$ to a finite set of ordinals below ω_α, and because h is surjective, every ordinal below ω_α belongs to one of these finite sets of ordinals. So $\{h``(f^{-1}``\{x\}) : x \in X\}$ is a set of finite sets of ordinals below ω_α whose sumset is ω_α, so it must be of size $\aleph_\alpha$. (*) However, we claimed more than this. We now need an explicit bijection between this set and the ordinals below ω_α so that we can compose it with $\lambda x \in X.(h``(f^{-1}``\{x\}))$ to get a surjection from X to the ordinals below ω_α.

   To find this explicit bijection observe that since $\{h``(f^{-1}``\{x\}) : x \in X\}$ covers the whole of ω_α its order type in the lexicographic order of finite sequences of ordinals cannot be less than ω_α, since ω_α cannot be the union of fewer than ω_α finite sets. It is the union of $|\{h``(f^{-1}``\{x\}) : x \in X\}|$ many finite sets, and this cardinal is an aleph so it must be at least $\aleph_\alpha$. Therefore applying the algorithm that compares the lengths of wellorderings to ω_α and $\{h``(f^{-1}``\{x\}) : x \in X\}$ will result in a canonical embedding $\omega_\alpha \to \{h``(f^{-1}``\{x\}) : x \in X\}$ (rather than the other way round). If this injection is onto, we are done. If not we have an injection $\omega_\alpha \to \{h``(f^{-1}``\{x\}) : x \in X\}$ and an injection the other way because there is a canonical (i.e., uniform in α) bijection $\omega_\alpha \longleftrightarrow (\omega_\alpha)^{<\omega}$. Then we use the construction of the Schröder-Bernstein theorem to obtain the explicit bijection that we need.

2. Our second construction elaborates a surjection $X \twoheadrightarrow \omega_\alpha$ into a surjection $\mathcal{P}(X) \twoheadrightarrow \omega_{\alpha+1}$

   A surjection $g_\alpha : X \twoheadrightarrow \omega_\alpha$ gives rise to a map $h_\alpha : \mathcal{P}(X) \twoheadrightarrow \omega_{\alpha+1}$ as follows. g_α will lift to a surjective map $\mathcal{P}(X) \twoheadrightarrow \mathcal{P}(\omega_\alpha)$. There is a canonical bijection—uniform in α—between ω_α and $\omega_\alpha \times \omega_\alpha$, so that every subset of ω_α can be thought of as a binary relation on ω_α. Send each such binary relation to its length if it is a worder and to 0 if it is not. This gives us $h_\alpha : \mathcal{P}(X) \twoheadrightarrow \omega_{\alpha+1}$ as claimed.

   This construction does not exploit f at all: it is merely the standard way of proving that $\aleph^*(|Y|) \leq^* 2^{|Y \times Y|}$.

3. Finally we show how to extract a surjection $\mathcal{P}(X) \twoheadrightarrow \omega_\lambda$ from a family $\langle g_\alpha : \alpha < \lambda \rangle$ of surjections $X \twoheadrightarrow \omega_\alpha$.

   For $\beta < \omega_\lambda$, let μ_β be the first g_α such that $\beta \in$ range of g_α. Let $X_\beta = \{x \in X : \mu_\beta(x) = \beta\}$, and let B be the set of all X_β. B is a homomorphic image of the ordinals below ω_λ, so $|B| \leq \aleph_\lambda$. Although there is no obvious way of excluding the possibility that there should be $\beta \neq \gamma$ with $X_\beta = X_\gamma$, we do at least know that this cannot happen if $\mu_\gamma = \mu_\beta$ for then X_β and $X_{\beta'}$ are disjoint so they are at least distinct. This means that for each $\alpha < \lambda$, all the X_β for $\omega_\alpha < \beta < \omega_{\alpha+1}$ are distinct. So $|B| \geq \aleph_\alpha$ for all $\alpha < \lambda$, so $|B| = \aleph_\lambda$ precisely. That is to say, if we order B, the set of all the B_β with $\beta < \omega_\lambda$ by their subscripts "at first appearance", it will be of length precisely ω_λ. Now we can get an injection $\omega_\lambda \hookrightarrow \mathcal{P}(X)$ by sending each $\beta < \omega_\lambda$ to the βth member of B. This in turn gives us a surjection $h : \mathcal{P}(X) \twoheadrightarrow \omega_\lambda$, by sending to 0 every subset of X not in B.

This construction does not exploit f either. (Nor does it use choice).

To complete the proof we now observe that if we start with an explicit surjection $X \twoheadrightarrow \omega$, the above constructions will give us an explicit transfinite sequence of g_αs as above. This ensures that $\aleph^*(|X|)$ (the cardinal of the least ordinal not the length of a prewellordering of X) cannot be successor (because of the first and second constructions) and cannot be limit (because of the second and third constructions). $\square$

The chief interest of this little result may lie perhaps not in the news that the class of finite sets is closed under yet another relation, and that this large cardinal property of $\aleph_0$ does not apparently generalise to all strongly inacessible cardinals ('*' marks the spot in the first construction where finiteness is exploited) nor even in the fact that the proof is not constructive (it is constructive in the sense that it embodies an algorithm that, were it ever (*per impossibile* to be given an infinite X and a finite-to-one map $f : \mathcal{P}(X) \twoheadrightarrow X$, would emit an unbounded stream of g_αs as above, though not in the sense that it embodies an algorithm that, presented with X and a finite-to-one map $f : \mathcal{P}(X) \twoheadrightarrow X$, will emit an injection from X into ω) but rather from the observation that to prove that the class of finite sets is closed under a fairly straightforward operation we apparently need an induction over all ordinals: the third construction exploited above relies on the axiom scheme of replacement. If no proof can be found that doesn't use replacment, one will start wondering whether there might not be models of Zermelo set theory (without choice of course) harbouring counterexamples. (A similar reflection arises in connection with Specker's proof, in [S], that if $x > 5$ then $2^x \not\leq x^2$). Such models might be objects of interest.

Bibliography

[S] E. Specker: Verallgemeinerte Kontinuumshypothese und Auswahlaxiom, Archiv der Mathematik 5 (1954), 332-337.

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