ZF + “Every set is the same size as a wellfounded set”

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Abstract

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Let ZFB be ZF + “every set is the same size as a wellfounded set”. Then the following are true.

> *Every sentence true in every (Rieger-Bernays) permutation model of a model of ZF is a theorem of ZFB. (i.e., ZFB is the theory of Rieger-Bernays permutation models of models of ZF)*
>
> *ZF and ZFAFA are both extensions of ZFB conservative for stratified formulæ.*
>
> *The class of models of ZFB is closed under creation of Rieger-Bernays permutation models.*

In this short note we look at the fertile and underregarded work of the late Jean Coret on ZF and illfounded sets, and extract some new and topical results from it. It is a pleasure to start it by thanking Randall Holmes for helpful comments he made on earlier versions of it.

Introduction

B is the axiom that says that every set is the same size as a wellfounded set. (This is Coret’s notation in [1964]). ZF includes the axiom of foundation but not the axiom of choice. ZFB is ZF with the axiom of foundation weakened to B. ZFAFA is ZF with foundation replaced by the Forti-Honsell antifoundation axiom, which says that every accessible pointed digraph is the $\in$-picture of a unique set (see [1984]). We are not assuming AC here: B is provable in ZFC-minus-foundation. If there were an illfounded set not the same size as any wellfounded set, then, by AC, it would have to be bigger than every wellfounded set, and the set of von Neumann ordinals that can be embedded in it would be the class of all ordinals.

The reader is assumed to know what a stratified formula is. We denote structures by characters in calligraphic font, using the corresponding upper-case Roman letters to denote the carrier set of the structure. If $\mathcal{M} = \langle M, \in \rangle$ is a structure for the language of Set Theory, and π is a permutation of M, we say that $\mathcal{M}^\pi = \langle M, \in_\pi \rangle$ (where $x \in_\pi y$ iff $x \in \pi(y)$) is a **permutation model** of $\mathcal{M}$. We say a formula ϕ is invariant iff every permutation model of a model of ϕ is also a model of ϕ. We speak similarly of invariant theories.

In this connection the idea of a **setlike** permutation is important. π is a setlike permutation of $\mathcal{M}$ if for all $n \in \mathbb{N}$, the restriction of $j^n(\pi)$ to M is a permutation of M, where $j(f)$ is $\lambda x. f``x$.

A $\mathcal{P}$-embedding is an embedding f satisfying the extra condition that if $x \subseteq f(y)$ then x, too, is a value of f. ("No new subsets") We write '$\mathcal{M} \prec_\mathcal{P} \mathcal{M}'$' to mean that that $M \subseteq M'$ and the inclusion embedding is a $\mathcal{P}$-embedding.

Results

The following observation is essentially due to Coret, though not in this form or in this generality.

Remark 1 *Let $\mathcal{M} \prec_\mathcal{P} \mathcal{M}'$ both be models ZF without foundation. Suppose that $\mathcal{M}' \models$ everything in M' is the same size as something in M. Then the inclusion embedding is elementary for stratified formulæ.*

Proof:

In the proof by induction on quantifiers and connectives the only case that involves any hard work is the existential quantifier, so let's concentrate on that. We are given that $\mathcal{M}' \models (\exists y)\phi(\vec{x}, y)$ where ϕ is stratified and the $\vec{x}$ are in M. We seek $y \in M$ witnessing the quantifier. We will a lemma due to Coret which says that if $\phi(x_1 \ldots x_n)$ is a stratified formula, and π a permutation, then $\phi(x_1 \ldots x_n) \longleftrightarrow \phi(j^{t_1}(\pi)(x_1) \ldots j^{t_n}(\pi)(x_n))$ where t_i is the type of x_i in ϕ. To keep things readable, let us suppose there are only two x variables, that y is of type 5, and that x_1 is of type 2 and x_2 of type 4. If there is such a y in M then we are done, so let us assume that there is none, and fix y to be a witness in M'.

To invoke Coret's lemma we must find a setlike permutation π such that $(j(\pi))(x_1) = x_1$, $(j^3(\pi))(x_2) = x_2$ and $(j^5(\pi))(y)$ is in M. To find such a π we must think of the action of π on the things in $\bigcup^2 x_1 \cup \bigcup^4 x_2$ (which we will abbreviate to A) and on $\bigcup^5 y$. π must fix everything in A and must send everything in $\bigcup^5 y \setminus A$ to something in M. Notice that the condition that $\mathcal{M}'$ be a $\mathcal{P}$-extension of $\mathcal{M}$ ensures that $\bigcup^5 y$ is not a subset of A. The other condition ensures that $\bigcup^5 y \setminus A$ is the same size as some element Y of M. Since $M \models ZF$ we can take Y to be disjoint from A. It is straightforward to extend the bijection between Y and $\bigcup^5 y \setminus A$ to a permutation fixing everything in A. (Here we use the fact that $\mathcal{M}' \models ZF$.) This permutation is the one we want.

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Notice that if $\mathcal{M} \models ZFB$ it is not just an end-extension but also a $\mathcal{P}$-extension of its wellfounded part. It is this fact that will make remark 1 so useful. Indeed a special case of this scenario is an illustration of why we really do need the assumption that every set in $\mathcal{M}'$ be the same size as a set in $\mathcal{M}$, We start with an arbitrary model $\mathcal{M}$ of ZF (with foundation). Add a lot of Quine atoms by a Rieger-Bernays permutation, which does not change the truth-value of any stratified formula, and do a Fraenkel-Mostowski construction to make choice false. (See Forster [2003] in press). If the wellfounded part of the Fraenkel-Mostowski model were a substructure of it elementary for stratified formulæ then the original would not have been a model of choice. The point is that in the FM model not every set is the same size as a wellfounded set. The remaining assumption in the theorem is that $\mathcal{M}$ and $\mathcal{M}'$ are models of ZF rather than anything weaker. This can be weakened, but a discussion of these matters will have to await Forster, Kaye and Mathias [20xx]

COROLLARY 2 *Every sentence true in all permutation models of all models of ZF is a theorem of ZFB.*

Proof: By the completeness theorem for stratified formulæ in Forster [1990] stratified formulæ are those preserved under permutation models, so the stratified theorems of ZF are precisely those true in all permutation models. But every permutation model of a model of ZF is a model of ZFB.

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COROLLARY 3 *ZF is an extension of ZFB conservative for stratified formulæ.*

Proof: Let ϕ be a stratified theorem of ZF, and $\mathcal{M}$ an arbitrary model of ZFB. Then ϕ is true in the wellfounded part of $\mathcal{M}$, (since that is a model of ZF). But by remark 1, the wellfounded part of $\mathcal{M}$ is a substructure elementary for stratified formulæ whence $\mathcal{M} \models \phi$.

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COROLLARY 4 *ZFAFA is an extension of ZFB conservative for stratified formulæ.*

Proof:

Let $\mathcal{M}$ be an arbitrary model of ZFB. We will obtain from it a model of ZFAFA satisfying the same stratified formulæ.

The wellfounded part $\mathcal{W}$ of $\mathcal{M}$ is a model of ZF. By remark 1 it satisfies the same stratified formulæ as $\mathcal{M}$.

Now Forti-Honsell in [1984] prove that given $\mathcal{W} \models ZF$ there is (a unique—though we do not make use of the uniqueness) $\mathcal{W}' \models ZFAFA$ with the same wellfounded sets as $\mathcal{M}$, and the illfounded sets of this model are all the same size as wellfounded sets. This $\mathcal{W}'$ is the desired model.

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Remark 5 *ZFB is invariant.*

Proof: Suppose $\mathcal{M} \models ZFB$ and τ is a permutation of M. We must show that $\mathcal{M}^\tau \models B$. (It is standard that in these circumstances $\mathcal{M}^\tau$ satisfies all the other axioms of ZFB). Consider an element x of $\mathcal{M}^\tau$. It has a *set* of members—in the sense of $\mathcal{M}$. This set—$\tau(x)$—is the same size as some wellfounded set y of $\mathcal{M}$, since $\mathcal{M} \models B$. Let i be the obvious recursively defined map from the wellfounded sets of $\mathcal{M}$ into $\mathcal{M}^\tau$. (set $i(w) =: \tau^{-1}(i``w)$, for all w in the wellfounded part of $\mathcal{M}$). Then $i(y)$ is a wellfounded set of $\mathcal{M}^\tau$. We claim that $\mathcal{M}^\tau$ believes $i(y)$ to be the same size as x. But $\mathcal{M}^\tau \models |x| = |i(y)|$ iff $\mathcal{M} \models |\tau(x)| = |\tau(i(y))|$. But this is certainly true, because, $\tau(i(y)) = i``y$, and $|i``y| = |y|$, and $|y| = |\tau(x)|$ by our original choice of y.

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Corollary 6 *ZFB is the theory of all permutation models of models of ZF.*

That is to say: if we close the class of models of ZF under the Rieger-Bernays permutation model construction we get more models, and accordingly a weaker theory of all those models. That weaker theory turns out to ZFB.

Concluding remarks

It would be interesting to know whether similar results hold for Zermelo set theory or KF. These issues are artificially simple when raised with respect to ZF. Remark 1 needs the permutation of which it treats to be setlike, and the axiom scheme of replacement ensures that all permutations are setlike. Without replacement results like the above will be much harder to obtain, although proofs of versions of remark 1 for weaker systems like those in Mathias [2001] can be had. These and other matters will be properly treated in Forster, Kaye and Mathias [20xx]

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