Why the Sets of NF do not form a Cartesian-closed Category

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This is an old result, and was first published by McLarty [2] relatively recently. It was discovered independently at least three times – to my certain knowledge – and quite possibly at least five times since my guess would be that Dana Scott and Solomon Feferman discovered proofs in addition to the three proofs known to me; I know Randall Holmes did, and when I met Edmund Robinson in about 1980 the first question he asked me – on learning that I studied NF – was whether or not the category of sets in NF was cartesian-closed. My reply ("What is a cartesian closed category?") led to my first lesson in category theory and to Edmund and I answering the question together.

Of these independent discoveries McLarty's has a special significance. Noticing that a given mathematical insight is important is itself a – separate[1] – mathematical insight, and even if Colin was not the first to see that the sets of NF do not form a cartesian-closed category, he certainly was the first to see that this failure mattered. In any case my purpose here is not to score points about priority, but rather to exhibit a proof which is quite different from Colin's. I have just cobbled it together as a result of a question asked me by Peter Lumsdaine, and thanks go to Peter for the stimulus. My guess is that it is the same proof Edmund and I discovered, tho' i might be wrong. There are evidently at least three proofs – since Holmes assures me that the proof he discovered is different from McLarty's and mine – and if there are three there may be more.

I start from the two uncontroversial assumptions

1. For the category of sets of a theory to be cartesian closed it is necessary for the theory to believe that the graph of the function

$$\texttt{curry}\colon (A \times B) \to C. \quad \to . \quad A \to (B \to C)$$

is a set (at least locally, in the sense that its restriction to any set is a set)[2];

[1] This is a slightly racier version of Dedekind's argument for the axiom of infinity according to which each thing is different from the concept of that thing.

[2] Thanks to Peter Johnstone for confirming that this is the case: I know no Topos theory!

2. In any sensible pairing function for NF, the expression '$x = \langle y, z \rangle$', when written out in primitive notation, must be stratified with 'y' and 'z' having the same type, and 'x' having a type which is not lower then the type of 'y' and 'z'.[3]

In NF the qualification at the end of item (1) makes no difference, since there is a universal set and the graph of `curry` local to it will be the graph of `curry` itself.

If the graph of `curry` is a set then in particular so is the graph (call it f_1) of the function that for each x sends $(\{\emptyset\} \times \{\emptyset\}) \to x$ to $\{\emptyset\} \to (\{\emptyset\} \to x)$. $\{\emptyset\} \to x$ is one type higher than x so – by NF comprehension – the graph (call it f_2) of the function sending $\{x\}$ to $(\{\emptyset\} \times \{\emptyset\}) \to x$ is a set. By the same token $\{\emptyset\} \to (\{\emptyset\} \to x)$ is two types higher than x, and – by NF comprehension again – the graph (call it f_3) of the function sending $\{\emptyset\} \to (\{\emptyset\} \to x)$ to $\{\{x\}\}$ is also a set.

Then the composition $f_3 \cdot f_1 \cdot f_2$ sends $\{x\}$ to $\{\{x\}\}$. This immediately gives us the graph of the singleton function as a set and this is known to be impossible in NF. ■

(Another way of characterising cartesian-closed categories is by the presence of an evaluation function $ev : (A \to B) \times A \to B$. Naturally a similar exercise will show that the graph of this function cannot be a set.)

How serious is this breakdown? It may be much less serious than one thinks. NFistes have known for a long time that whenever a desired result P – familiar from ZF – fails in NF then the proof method that gave rise to P can be tweaked to prove a variant P' which

(i) is equivalent to P in ZF; and

(ii) differs from P in having a T function inserted judiciously, or in having some occurences of a variable 'X' replaced by 'ι"X', or some similar modification.

(iii) P' discharges many of the functions of P and life goes on roughly as normal. One thinks of the theorem $|\iota``x| < |\mathcal{P}(x)|$ which does duty for Cantor's theorem in NF. Perhaps the failure of cartesian-closedness of NF will turn out to be similarly un-concerning.

[3]There is a fairly detailed discussion of this point in [1].

References

[1] Forster, T.E. "Implementing Mathematical Objects in Set Theory" Logique et Analyse **50** No.197 (2007). Also available from `www.dpmms.cam.ac.uk/~tef10/pairs.pdf`

[2] McLarty, C. "Failure of cartesian closedness in NF". *Journal of Symbolic Logic* **57** 1992 pp. 555−6.

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This is an old document (the previous .pdf output was dated october 14th 2007) and the only changes i have made are copy-editing and updating of bibliographic entries.