Workout 26 p 525

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February 22, 2015

$F \subseteq \mathcal{P}(A)$, and $U = \{X \subseteq A : (\forall S \in F)(S \subseteq X)\}$. Show $\bigcup F = \bigcap U$.

Remember not to panic! The first thing to do is make sure you stare at these things long enough to understand exactly what is being said. We put into U all the things that extend ("$\supseteq$") everything in F. This immediately gives us $\bigcup F \subseteq \bigcap U$. OK, this is beco's if $x \in \bigcup F$ then there is $S \in F$ with $x \in S$. But any such S is included in everything in U, so x belongs to everything in U—which is to say $x \in \bigcap U$. So we have proved $x \in \bigcup F \to x \in \bigcap U$. But x was arbitrary, so we have proved $(\forall x)(x \in \bigcup F \to x \in \bigcap U)$—which is to say $\bigcup F \subseteq \bigcap U$.

The other direction is a weeee bit harder. Suppose $x \in \bigcap U$, which is to say it belongs to everything in U, so it belongs to everything that passes the membership test for U. Now $U = \{X \subseteq A : (\forall S \in F)(S \subseteq X)\}$, so we infer $(\forall X)((\forall S \in F)(S \subseteq X)) \to x \in X)$.

Now if x belongs to all X satisfying $(\forall S \in F)(S \subseteq X))$ then it must certainly belong to the $\subseteq$-least of them. What might that be? Clearly the union of all those S's.