1. Countable Ordinals, Proof theory and Fast-growing Functions . . . .

Thomas Forster

A good point of departure for this essay would be the Part II ST&L exercise in which the student is invited to show how, for every countable ordinal, a subset of $\Re$ can be found that is wellordered to that length in the inherited order. The obvious way to do this involves induction on countable ordinals and leads swiftly to the discovery of fundamental sequences. These can be put to work immediately in the definition of hierarchies of fast-growing functions. This leads in turn to the Schmidt conditions, which the student should explain carefully. There is a wealth of material on how proofs of totality for the faster-growing functions in this hierarchy have significant—indeed *calibratable*—consistency strength. One thinks of Goodstein's function and Con(PA), or of Paris-Harrington. There is plenty here from which the student can choose what to cover.
The Doner-Tarski hierarchy of functions (addition, multiplication, exponentiation . . .) invites a transfinite generalisation and supports a generalisation of Cantor Normal form for ordinals. Nevertheless, the endeavour to notate ordinals beyond ϵ_0 does not use those ideas, but rather the enumeration of fixed points: such is the *Veblen hierarchy*. From this one is led to the impredicative Bachmann notation, with Ω and the ϑ function.

Relevant Courses

Essential:
Part II Set Theory and Logic

References

Schwichtenberg-Wainer Proofs and Computations Cambridge University Press (go to `http://www.cambridge.org/us/academic/subjects/mathematics/logic-categories-and-sets/proofs-and-computations`)
`http://www.math.ucsb.edu/~doner/articles/Doner-Tarski.pdf`
`http://www.dpmms.cam.ac.uk/~tf/ordinalsforwelly.pdf`

2. Wellquasiorders and Betterquasiorders . . . .

Dr. T. E. Forster

A well-quasi-order is a reflexive transitive relation with no infinite descending chains and no infinite antichains. Although this definition may not sound particularly natural there are many natural examples, at least one of which is famous: the theorem of Seymour and Robertson that finite graphs under the graph minor relation form a WQO (although this is far beyond the scope of a Part III essay!) There is Laver's theorem that the isomorphism types of scattered total orders (orders in which the rationals cannot be embedded) form a WQO. A proper treatment of this proof would almost be enough by itself for an essay. Kruskal's theorem states that finite trees with nodes labelled with elements of a WQO are also WQO-ed. This has a very striking finitisation which is associated with a very fast-growing function and a consistency proof for PA.
The class of WQO's lacks certain nice closure properties and the project to patch this up leads to a concept of *Better*-quasi-ordering. The class of BQOs is algebraically nicer.
These combinatorial ideas have wide ramifications in Graph Theory, Logic and Computer Science (lack of infinite descending chains is always liable to be connected with termination of processes) and the area has a good compact literature and some meaty theorems. Recommended for those of you who liked the Logic course and the Combinatorics course.
A Big Plus for this topic is that there is no textbook! (The setter dreams of writing one and the current draught is linked below) There is a wealth of literature. Interested students should consult the setter.

Relevant Courses

Essential: None
Useful: Combinatorics, Logic and Set Theory

References

`http://www.dpmms.cam.ac.uk/~tf/BQObok.pdf`

3. Nonstandard Analysis .... Dr. Forster

Description of Essay

Developments in C20 Logic have made possible a rigorous treatment of infinitesimals, and thereby opened the door to a natural rational reconstruction of C17 calculus. However the interest of this material is not just historical but emphatically mathematical. Although the C20 discoveries that started this renascence lie in Logic, the Logic involved is not terribly recondite, and the material is accessible also to people in Analysis; it deserves to be more widely known.
The *locus classicus* for C20 Nonstandard Analysis is of course Abraham Robinson *op cit*, but nowadays there are are other approaches to infinitesimals that have become available. Rather than cover all of them the student may prefer to concentrate on just one.
There is a wealth of available literature, most of the essentials of which is listed below.

Relevant Courses

Essential: Part II Logic and Set Theory or equivalent.
Familiarity with Undergraduate Analysis is essential.

References

[1] John Bell "A Primer of Infinitesimal Analysis" CUP
[2] H. Jerome Keisler "Elementary Calculus: An Infinitesimal approach". `http://www.math.wisc.edu/~keisler/calc.html`
[3] André Pétry "Analyse Infinitesimale—Une Presentation Nonstandard" Céfal 2010
[4] Edward Nelson "Internal set theory" `https://web.math.princeton.edu/~nelson/books/1.pdf`

[5] Sergio Albeverio, Raphael Høegh-Krohn, Jens Erik Fenstad and Tom Lindstrøm, Nonstandard methods in stochastic analysis and mathematical physics
Bull. Amer. Math. Soc. (N.S.) Volume 17, Number 2 (1987), 385-389.

[6] Cutland, Neves, Oliveira and Pinto "Developments in Nonstandard Mathematics" Longman 1995

[7]; J. Avigad, J. Helzner, Transfer principles for intuitionistic nonstandard arithmetic, Arch. Math. Logic. Archive for Mathematical Logic August 2002, Volume 41, Issue 6, pp 581-602

[8] An Introduction to Nonstandard Real Analysis, Volume 118 (Pure and Applied Mathematics) by Albert E. Hurd, Peter A. Loeb. Academic Press

[9] Non-standard Analysis. By Abraham Robinson. Princeton University Press, 1974.