# Discussion answers to old tripos questions

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## 1988:6:9E (maths tripos)

The Master asked $2n + 1$ people and got $2n + 1$ different answers. Since the largest possible answer is $2n$ and the smallest is 0, there are in fact precisely $2n + 1$ possible answers and that means he has got every possible answer from 0 up to $2n$ inclusive.

Think about the person who shook $2n$ hands. This person shook hands with everyone that they possibly could shake hands with : that is to say everyone except their spouse. So everybody except their spouse shook at least one hand. So their spouse shook no hands at all. Thus the person who shook $2n$ hands and the person who shook 0 hands are married. Henceforth disregard these two people and their handshakes and run the same argument to show that the person who shook $2n - 1$ hands and the person who shook 1 hands are married. And so on.

Where does this get us? It tells us, after $n$ iterations, that the person who shook $n+1$ hands and the person who shook $n - 1$ hands are married. So what about the person who shook $n$ hands, the odd man out? Well, it must be the odd *woman* out, because the only person of whom the Master asks this question who isn't married to another person of whom the Master asks this question is his wife.

Let's name people (other than the Master) with the number of hands they shook. (This is ok since they all shook different numbers of hands.) $2n$ didn't shake hands with its spouse, or itself, and there are only $2n$ people left, so it must have shaken hands with all of them, in particular with the Master. Correspondingly 0 didn't shake hands with anyone at all, so it certainly didn't shake hands with the Master. We continue reasoning in this way, about $2n - 1$ and 1. $2n - 1$ didn't shake hands with itself or its spouse or with 0, and that leaves only $2n - 1$ people for it to shake hands with and since it shook $2n - 1$ hands it must have shaken all of them, so in particular it must have shaken hands with the Master. Did 1 shake hands with the Master? No, because 1 shook only one hand, and that must have been $2n - 1$'s. And so on. The people who shook the Master's hand were $2n$, $2n - 1$, $2n - 2 \ldots n + 1$ and the people who didn't were $1, 2, 3, \ldots n - 1$. [or do i mean '$n$'?] And of course, the Master's wife. So he shook $n$ hands.

See also

`https://www.dpmms.cam.ac.uk/~tf/cam_only/keeping_out_of_mischief.htm`

## 1988:6:10E (maths tripos)

Let $R$ be a relation on a set $X$. Define the reflexive, symmetric and transitive closures $r(R)$, $s(R)$ and $t(R)$ of $R$. Let $\Delta$ be the relation $\{\langle x, x\rangle : x \in X\}$. Prove that

1. $R \circ \Delta = R$
2. $(R \cup \Delta)^n = \Delta \cup (\bigcup_{i \leq n} R^i)$ for $n \geq 1$
3. $tr(R) = rt(R)$.

Show also that $st(R) \subseteq ts(R)$.

If $X = \mathbb{N}$ and $R = \Delta \cup \{\langle x, y\rangle : y = px \text{ for some prime } p\}$ describe $st(R)$ and $ts(R)$.

The reflexive (symmetric, transitive) closure of $R$ is the intersection of all reflexive (symmetric, transitive) relations of which $R$ is a subset.

1. $R\Delta$ is $R$ composed with the identity relation. $x$ is related to $y$ by $R$-composed-with-$S$ if there is $z$ such that $x$ is related to $z$ by $R$, and $z$ is related to $y$ by $S$. Thus $R\Delta = R$. (I would normally prefer to write '$R \circ \Delta$' here, using a standard notation for composition of relations: '$\circ$')

2. It is probably easiest to do this by induction on $n$. Clearly this is true for $n = 1$, since the two sides are identical in that case. Suppose it is true for $n = k$.

$$(R \cup \Delta)^k = \Delta \cup (\bigcup_{1 \leq i \leq k} R^i)$$

$(R \cup \Delta)^{k+1} = (R \cup \Delta)^k \circ (R \cup \Delta)$. By induction hypothesis this is

$$(\Delta \cup (\bigcup_{1 \leq i \leq k} R^i)) \circ (R \cup \Delta)$$

Now $(A \cup B) \circ (C \cup D)$ is clearly $(A \circ C) \cup (A \circ D) \cup (B \circ C) \cup (B \circ D)$ and applying this here we get

$$(\Delta \circ R) \cup (\Delta \circ \Delta) \cup ((\bigcup_{1 \leq i \leq k} R^i) \circ R) \cup ((\bigcup_{1 \leq i \leq k} R^i) \circ \Delta)$$

Now $\Delta \circ R$ is $R$; $\Delta \circ \Delta$ is $\Delta$; $(\bigcup_{1 \leq i \leq k} R^i) \circ \Delta$ is $\bigcup_{1 \leq i \leq k} R^i$ and $(\bigcup_{1 \leq i \leq k} R^i) \circ R$ is $(\bigcup_{1 < i \leq k+1} R^i)$ so we get

$$R \cup \Delta \cup (\bigcup_{1 < i \leq k+1} R^i) \cup (\bigcup_{i \leq k} R^i)$$

which is

$$\Delta \cup \bigcup_{1 \leq i \leq k+1} R^i$$

3. The transitive closure of the reflexive closure of $R$ is the transitive closure of $R \cup \Delta$ which is $\bigcup_{n \in \mathbb{N}} (R \cup \Delta)^n$ which (as we have—more-or-less—just proved) is $\Delta \cup (\bigcup_{i \in \mathbb{N}} R^i)$ which is the reflexive closure of the transitive closure of $R$.

$s$ is *increasing* so $R \subseteq s(R)$. $t$ is *monotone*, so $t(R) \subseteq t(s(R))$. But the transitive closure of a symmetrical relation is symmetrical so $t(R) \subseteq t(s(R))$ implies $s(t(R)) \subseteq t(s(R))$ as desired.

Finally if $X = \mathbb{N}$ and $R = \Delta \cup \{\langle x, y \rangle : y = px$ for some prime $p\}$ then $st(R)$ is the relation that holds between two numbers when they are identical or one is a multiple of the other, and $ts(R)$ is the universal relation $\mathbb{N} \times \mathbb{N}$.

## 1990:1:9

Peter Dickman's model answer

We are asked to use generating functions to prove that:

$$c_n = \frac{1}{n} \binom{2n-2}{n-1}$$

where $c_n$ is the number of binary trees with $n$ leaves (NB not $n$ vertices) where no vertex has precisely one descendent. Now the formula given is remarkably similar to the one for Catalan numbers – which were introduced in the section of the course concerned with generating functions. So these may well be useful in answering this question.

Recall that for Catalan numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

so it is (hopefully) clear that $C_n = c_{n+1}$.

To use generating functions it is necessary to find a recurrence relation...

Consider trees of the form described in the question. Clearly, any such tree which has more than one leaf can be viewed as being composed of two trees joined together by a single (new) root vertex, whose descendents are the two roots of the component smaller trees. Now the sum total of leaves in these sub-trees will be the same as the number of leaves in the composite; and each tree will have at least one leaf. So, the number of trees with some given number of leaves can be determined by considering all of the ways such a tree can be split into left & right subtrees, and the parts combined together.

It follows that:

$$\forall n \geq 1 : c_{n+1} = c_1 c_n + c_2 c_{n-1} + c_3 c_{n-2} + \ldots + c_n c_1$$

Note that we have $n \geq 1$ in the above, because the equation is giving us an expression for $n + 1$. The recurrence only holds for the trees with two or more leaves (as we assumed that the root had two descendents).

Also we know that $c_1 = 1$ by inspection.

Note that I've written this out for the case $n + 1$ not, as I would normally do, the case $n$ because it makes everything neater later. The result can be achieved from the $n$ case but is a bit messier. The only hint I can give as to how to tell that this is helpful **in advance** is that we already knew that there was an "off by one" effect present in this question.

Now, let us consider $d_n = c_{n+1}, \forall n \geq 0$. Then we have that:

$$\forall n \geq 1 : d_n = d_0 d_{n-1} + d_1 d_{n-2} + \ldots + d_{n-1} d_0$$

Now, if we define $d_k = 0, \forall k < 0$ then we have that

$$\forall n \geq 1 : d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i} = \sum_{i=0}^{\infty} d_i d_{n-1-i}$$

Now, the generating function for the $d_n$, called $D(z)$ say, has the property that the coefficient of $z^n$ in $D(z)$ is $d_n$. So we have that:

$$[z^n]D(z) = \begin{cases} 1 & \text{if } n = 0 \\ \sum_{i=0}^{\infty} d_i d_{n-1-i} & \text{otherwise} \end{cases}$$

Whence we derive:

$$\begin{aligned}
D(z) &= \sum_{n=0}^{\infty} d_n z^n \\
&= 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} d_i d_{n-1-i} z^n \\
&= 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} d_i d_{n-1-i} z^i z^{n-1-i} z \\
&= 1 + z \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} d_i d_{n-1-i} z^i z^{n-1-i} \\
&= 1 + z \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} d_i z^i \cdot d_{n-1-i} z^{n-1-i} \\
&= 1 + z(D(z))^2
\end{aligned}$$

since the penultimate line is a convolution.

This is a formula we recognise from the Catalan numbers, so we proceed by following the same argument as in the lecture notes...

Reorganising this gives us:

$$z(D(z))^2 - D(z) + 1 = 0$$

Solving this we find that:

$$D(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

Since $d_n$ is non-negative $\forall n$ and since $\sqrt{1-4z}$ has only negative signs after the first term we can eliminate the form with an addition in and find:

$$D(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

Which, from a standard binomial identity leads us to:

$$D(z) = \frac{1 - \sqrt{1-4z}}{2z} = \sum_{k \geq 0} \frac{1}{k+1} \binom{2k}{k} z^k$$

So we find that:

$$d_n = [z^n]D(z) = \frac{1}{n+1} \binom{2n}{n}$$

However, $d_n = c_{n+1}, \forall n \geq 0$ therefore we have that:

$$\forall n \geq 1 : c_n = \frac{1}{n} \binom{2n-2}{n-1}$$

as required.

Note that the formula is obviously useless for $n = 0$ as it would give $c_0 = \infty$ so we clearly aren't being expected to worry about that case. However it might be worth pointing this out.

The second part of the question asks how many trees of the form considered, with $n$ leaves, have depth $n-1$. Again let's look for a recurrence relation. I'll skip through this fairly quickly... I suggest that you draw some pictures as you read through this. Be aware of the assumption that $n > 2$ in the following.

Consider the trees of this form, that have $n$ leaves and are of depth $n-1$, for arbitrary $n > 2$. Given such a tree, let the number of leaves at depth $n-1$ (ie the maximal depth) be $k$.

From such a tree we can construct $k$ distinct trees of depth $n$ which have $n+1$ leaves by taking one of the leaves at the $n-1$ level and replacing it with a vertex with two descendents, which are themselves leaves.

Now consider a tree of depth $n$ with $n+1$ leaves, satisfying the condition on numbers of descendents. Selecting any leaf at the maximal depth, its parent is at depth $n-1$ and, by the condition on numbers of descendents, this has another child at depth $n$. Replacing these two leaves and their parent vertex with a single leaf at depth $n-1$ we either construct a tree with $n$ leaves of depth $n-1$ (if we have removed the only pair of leaves at the maximal depth) or we have a tree of $n$ leaves of depth $n$.

However the depth of one of our trees must be strictly less than the number of leaves. Assume otherwise, ie that for some such tree, the number of leaves is less than or equal to the depth. Since each 'plucking' operation of the form described above reduces the number of leaves by one and the number of levels by at most one, we would be able to construct a tree with 2 leaves and depth of at least 2 – which is clearly impossible.

So, we have shown that each such tree has precisely 2 leaves at its terminal level, and that the only possible constructions are the $k$ variants of each of the trees of one smaller size. But $k$ is the number of leaves at the terminal level *i.e.* 2, so we have a doubling of the number of possible trees at each level. Given that there are $1 = 2^0$ trees with 2 leaves of depth 1, $2 = 2^1$ trees with 3 leaves of depth 2 and so forth we have that,

$$(\forall n \geq 2)(\exists\, 2^{n-2} \text{ trivalent trees with } n \text{ leaves and depth } n-1)$$

## 1990:1:11

Equivalence relations correspond to partitions. A PER $\langle X, R\rangle$ that fails to be an equivalence relation features elements $x \in X$ such that $\langle x, x\rangle \notin R$. Such elements are not related to anything at all, since if $x$ is related to $y$, then by symmetry $y$ is related to $x$ and by transitivity $x$ is related to $x$.

So we put on one side all the $x \in X$ such that $\langle x, x\rangle \notin R$, leaving behind a subset $X' \subseteq X$ consisting of all those elements related to themselves by $R$. What is the restriction of $R$ to this set? Clearly it is reflexive. Actually it is transitive and symmetrical as well, because $X' \times X'$ is transitive and symmetrical, and transitivity and symmetry are intersection-closed properties so $R \cap (X' \times X')$ will be transitive and symmetrical. So $R \cap (X' \times X')$ is an equivalence relation on $X'$.

How many PER's on a set with 4 elements?

There is one way of throwing away no elements, leaving 4. These can be either

| | |
|---|---|
| all in one piece | 1 |
| one singleton, one triple | 4 |
| two pairs | 3 |
| two singletons | 6 |
| all singletons | 1 |

There are 4 ways of throwing away one element, leaving 3. These can be either

| | |
|---|---|
| all in one piece | $1 \times 4 = 4$ |
| one singleton, one pair | $3 \times 4 = 12$ |
| three singletons, | $1 \times 4 = 4$ |

There are 6 ways of throwing away two elements, leaving 2. These can be either

| | |
|---|---|
| all in one piece | $1 \times 6 = 6$ |
| two singletons | $1 \times 6 = 6$ |

There are 4 ways of throwing away 3 elements leaving 1.

| | |
|---|---|
| This can be partitioned in only one way | $1 \times 4 = 4$ |
| | $\overline{52}$ |

Let us prove that $T$ is a PER. We first show that it is transitive. Suppose

(i) $\langle f, g\rangle \in T$ and

(ii) $\langle g, h\rangle \in T$. We want $\langle f, h\rangle \in T$.

By definition of $T$ we infer

(iii) $(\forall x_1, x_2 \in X)(\langle x_1, x_2\rangle \in R \to \langle f(x_1), g(x_2)\rangle \in S)$ and

(iv) $(\forall x_2, x_3 \in X)(\langle x_2, x_3\rangle \in R \to \langle g(x_2), h(x_3)\rangle \in S)$. (We have relettered variable to make life easier)

Now let $x_1$ and $x_3$ be two elements of $X$ such that $\langle x_1, x_3\rangle \in R$. We want to infer $\langle f(x_1), g(x_3)\rangle \in S$. $R$ is symmetrical so $\langle x_3, x_1\rangle \in R$ too. So, by transitivity, we have $\langle x_1, x_1\rangle \in R$. By (iii) we can infer $\langle f(x_1), g(x_1)\rangle \in S$. We now use (iv) on our assumption that $x_1$ and $x_3$ are two elements of $X$ such that $\langle x_1, x_3\rangle \in R$ to infer that $\langle g(x_1), h(x_3)\rangle \in S$. Finally, by transitivity of $S$, we infer that $\langle f(x_1), h(x_3)\rangle \in S$ as desired.

It is much easier to show that $T$ is symmetric. Suppose $\langle f, g\rangle \in T$ and let $x_1$ and $x_2$ be two elements of $X$ such that $\langle x_1, x_2\rangle \in R$. We want to infer $\langle f(x_1), g(x_2)\rangle \in S$. $R$ is symmetric, so we infer $\langle x_2, x_1\rangle \in R$, whence $\langle f(x_1), g(x_2)\rangle \in S$ as desired.

To show that $T$ is not in general reflexive, even if $R$ and $S$ both are, take $R$ to be the universal relation on $X$ and $S$ to be the identity relation on $Y$, where both $X$ and $Y$ have at least two members.

## 1993:11:11

$\langle A, \leq\rangle$ is a partially ordered set if

1. $(\forall x, y, z \in A)(x \leq y \to (y \leq z \to x \leq z)$ ($\leq$ is transitive)
2. $(\forall x, y \in A)(x \leq y \to (y \leq x \to x = y))$ ($\leq$ is antisymmetrical)
3. $(\forall x \in A)(x \leq x)$ ($\leq$ is reflexive)

In what follows we write '$x < y$' for '$x \leq y \wedge y \neq x$'

(a) If $\langle A, \leq\rangle$ is to form a totally ordered set then in addition $\leq$ must satisfy *connexity*.

$$(\forall x, y \in A)(x \leq y \vee y \leq x)$$

or equivalently $<$ must satisfy *trichotomy*

$$(\forall x, y \in A)(x < y \vee x = y \vee y < x)$$

(b) If $\langle A, \leq\rangle$ is to be wellfounded then in addition $<$ (which is the strict version of $\leq$, namely $\{\langle x, y\rangle : x \leq y \wedge x \neq y\}$) must satisfy *wellfoundedness*:

$$(\forall A' \subseteq A)(\exists x \in A')(\forall y \in A')(y \not< x)$$

(This *détour* via strict partial orders is necessary because no wellfounded relation can be reflexive.)

> (c) If $\langle A, \leq\rangle$ is to be a complete partially ordered set then one of the following conditions on $\leq$ must be satisfied, depending on what your definition of complete poset is:
>
> One definition is that every *subset* of $A$ must have a least upper bound in the sense of $\leq$. This is
>
> $$(\forall A' \subseteq A)(\exists x \in A)[(\forall y \in A')(y \leq x) \wedge (\forall z \in A)((\forall y \in A')(y \leq z) \rightarrow x \leq z)]$$
>
> ...or that every directed subset of $A$ has a least upper bound. $A'$ is a directed subset of $A$ if $(\forall x, y \in A')(\exists z \in A')(x \leq z \wedge y \leq z)$. (They probably don't mean that tho'.)

To show that the restriction of a partial order of $A$ to some subset $B$ of $A$ is a partial order of $B$ we have to check that $R \cap (B \times B)$ is reflexive transitive and antisymmetrical. Now $B \times B$ is reflexive and transitive, as is $R$; reflexivity and transitivity are intersection-closed properties, so $R \cap (B \times B)$ is reflexive and transitive. To verify antisymmetry we have to check that if $\langle x, y\rangle$ and $\langle y, x\rangle$ are both in $R \cap (B \times B)$ then $x = y$. But if $\langle x, y\rangle$ and $\langle y, x\rangle$ are both in $R \cap (B \times B)$ then they are both in $R$, and we know $R$ is antisymmetrical, whence $x = y$ as desired.

(A deeper proof can be obtained by noting only that all the clauses in the definition of partial order are universal. Any universal sentence true in $A$ is true in any subset of $A$. After all, a universal sentence is true as long as there is no counterexample to it. If $A$ contains no counterexamples, neither can any subset of $A$. This shows that a substructure of a total order is a total order and this is useful later on in the question ...)

**Z** (i) $\leq$ is a partial order of $\mathbf{Z}$. Indeed it is a total order. (ii) It isn't wellfounded (e.g.: no bottom element) nor (iii) is it a complete poset (e.g.: no top element).

Divisibility (i) is not a partial order because for any integer $n$, $n$ and $-n$ divide each other but are distinct, so it isn't antisymmetrical. (ii) The relation "$n$ divides $m$ but not *vice versa*" is wellfounded on $\mathbf{Z}$ however. If $X \subseteq \mathbf{Z}$, then its minimal elements under "$n$ divides $m$ but not *vice versa*" are precisely the minimal elements of $\{|n| : n \in X\}$ under "$n$ divides $m$ but not *vice versa*", and this relation, being a subset of a wellfounded relation (and $\leq$ *is* wellfounded on $\mathbb{N}$) is itself wellfounded. (iii) $\mathbf{Z}$ is not a complete poset under divisibility for the same reason as before.

$\mathbb{N}$ $\leq$ is a partial order of $\mathbb{N}$. Indeed it is a total order. It is also wellfounded but it is not a complete poset (as before)

Divisibility is a partial order on $\mathbb{N}$ but not a total order, it is wellfounded. This time we do get a complete poset, because everything divides 0.

$\mathbb{N}^+$ As for $\mathbb{N}$ except that it is not a complete poset (e.g.: no top element)

## 1994:10:11

The way to do part 2 is to stop trying to be clever and do it the easy way. Let $A_n$, $B_n$, $C_n$ be the number of valid strings in $\{A, B, C\}^n$ ending in $A$, $B$ and $C$ respectively. Clearly

$$C_{n+1} = A_n + B_n + C_n$$

and

$$A_{n+1} = B_{n+1} = B_n + C_n$$

This is because if the last character of a legal string is an $A$ or a $B$ then the penultimate character cannot be an $A$. We are not going to try to do anything clever like *derive* the equality we have been given, but we can at least confirm it! So let's try to simplify

$$2(A_{n+1} + B_{n+1} + C_{n+1}) + A_n + B_n + C_n$$

and hope that it simplifies to $A_{n+2} + B_{n+2} + C_{n+2}$.

Take out $B_{n+1} + C_{n+1}$ twice to give $A_{n+2} + B_{n+2}$, leaving $2A_{n+1} + A_n + B_n + C_n$. The last three terms add up to $C_{n+1}$, and $2A_{n+1} = A_{n+1} + B_{n+1}$ so this is $A_{n+1} + B_{n+1} + C_{n+1}$ which is $C_{n+2}$. Together with the $A_{n+2} + B_{n+2}$ this adds up to $v(n + 2)$ as desired.

Part 3 is 'A'-level maths that you remember from your crèche.

## 1995:5:4X (maths 1a)

Well, adapted from it!

```
fun f n = if n = 0 then 0 else g(f(n-1) + 1, 1) -1
and g(n,m) = f(f(n-1)) + m + 1;
```

What are the `ML` types of these two functions?
What are the running times of `f` and `g`?
By inspection we notice that $(\forall n \in \mathbb{N})(\mathtt{f}(n) = n)$, but we had better prove it! It's true for $n = 0$. For the induction step the recursive declaration tells us that

$\mathtt{f}(n + 1) = \mathtt{g}(\mathtt{f}(n) + 1, 1) - 1$ (by substituting $n + 1$ for $n$)

But $\mathtt{f}(n) = n$ by induction hypothesis so this becomes

$\mathtt{f}(n + 1) = \mathtt{g}(n + 1, 1) - 1$

Now, substituting $(n + 1)$ for $n$ and 1 for $m$ in the declaration for `g` we get

$\mathtt{g}(n + 1, 1) = (n + 1 - 1) + 1 + 1$

which is $n + 2$, giving $\mathtt{f}(n + 1) = n + 1$ as desired.

**(d)**

The mutual recursion gives us a pair of mutual recurrence relations:

A: $F(n) = G(f(n-1)+1, 1) + F(n-1)$

B: $G(n, m) = F(n-1) + F(\mathtt{f}(n-1)) + k$

where $F$ is the cost function for `f` and $G$ is the cost function for `g`.
Using $\mathtt{f}(n) = n$ we can simplify our recurrence relations as follows.

A′: $F(n) = G(n, 1) + F(n-1)$

B′: $G(n, m) = F(n-1) + F(n-1) + k$ whence

B″: $G(n, m) = 2 \cdot F(n-1) + k$

This gives

$$F(n) = F(n-1) + F(n-1) + F(n-1) + k$$

so $F(n)$ grows like $3^n$.

$G$ is exponential too. We have assumed that the cost of adding the second agument (‘$m$’) is constant, but altho’ this simplification will cause no problems it is a simplification nevertheless. Adding two arguments takes time proportional to the logarithm of the larger of the two. Fortunately the cost functions of these algorithms are so huge that an extra log or two will make no difference to the order.

## 1996:1:7

A partial ordering is a relation that is reflexive, antisymmetrical and transitive.

‘Topological sort’ is Compsci jargon for refining a partial ordering, which just means adding ordered pairs to a partial ordering to get a total ordering. The two partial orders of $\mathbb{N} \times \mathbb{N}$ that you have seen are the **pointwise product** ($\langle x, y\rangle \leq_p \langle x', y'\rangle$ iff $x \leq x' \wedge y \leq y'$) and the **lexicographic product** ($\langle x, y\rangle \leq_{lex} \langle x', y'\rangle$ iff $x < x' \vee (x = x' \wedge y \leq y')$). The second is clearly a refinement of the first. It is also clear that the lexicographic product $\mathbb{N} \times \mathbb{N}$ is not isomorphic to $\mathbb{N}$ in the usual ordering, since it consists of $\omega$ copies of $\mathbb{N}$. ($\omega$ is the length of $\mathbb{N}$ in its usual ordering: the length of $\mathbb{N} \times \mathbb{N}$ in the product ordering is therefore said to be $\omega^2$).

To get a refinement of the product ordering of $\mathbb{N} \times \mathbb{N}$ that is isomorphic to the usual ordering on $\mathbb{N}$ we notice that for a wellordering to be isomorphic to the usual ordering on $\mathbb{N}$ it is sufficient for each point to have only finitely many things below it (given that is also a wellordering, that is). Try $\langle x, y\rangle \leq \langle x', y'\rangle$ iff $(x + y) < (x' + y') \vee (x + y = x' + y' \wedge x \leq x')$. It’s a total order, each element has only finitely many things below it (so it’s isomorphic to the usual order on $\mathbb{N}$) and it refines the pointwise product ordering.

## 1996:1:8

The recurrence

$$R: \; w(n, k) = w(n - 2^k, k) + w(n, k-1)$$

can be justified as follows. Every representation of $n$ pfatz as a pile of coins of size no more than $2^k$ pfatz either contains a $2^k$ pfatz piece or it doesn’t. Clearly there are

$w(n, k-1)$ representations of $n$ pfatz as a pile of coins of size no more than $2^{k-1}$ pfatz so that's where the $w(n, k-1)$ comes from. The other figure arises from the fact that a representation of $n$ pfatz as a pile of coins of size no more than $2^k$ pfatz and containing a $2^k$ pfatz piece arises from a representation of $n - 2^k$ pfatz as a pile of coins of size no more than $2^k$.

Base case. $w(n, 0) = 1$. That should be enough.

To derive $w(4n, 2) = (n + 1)^2$, substitute $4n$ for $n$, and 2 for $k$ in $R$, getting

$$w(4n, 2) = w(4n - 2^2, 2) + w(4n, 1)$$

But this rearranges to

$$w(4n, 2) = w(4(n - 1), 2) + w(4n, 1)$$

$w(4n, 1)$ is $2n+1$, since we can have between 0 and $2n$ 2-pfatz pieces in a representation of $4n$. This gives

$$w(4n, 2) = w(4(n - 1), 2) + 2n + 1$$

This is a bit clearer if we write this as $f(n) = f(n-1)+2n+1$. This recurrence relation obviously gives $f(n) = (n + 1)^2$ as desired .

We can always get an estimate of $w(n, k)$ by applying equation $R$ recursing on $n$, and this works out quite nicely if $n$ is a multiple of $2^k$ because then we hit 0 exactly, after $n/(2^k)$ steps. Each time we call the recursion we add $w(n, k-1)$ (or rather $w(n-y, k-1)$ for various $y$) and clearly $w(n, k - 1)$ is the biggest of them. So $w(n, k)$ is no more than $n/(2^k) \cdot w(n, k - 1)$.

Finally, using $R$ with $2^{k+1}$ for $n$ again we get $w(2^{k+1}, k) = w(2^k, k) + w(2^{k+1}, k - 1)$. The hint reminds us that every representation of $2^k$ pfatz using the first $k$ coins gives rise to a representation of $2^{k+1}$ pfatz using the first $k + 1$ coins. Simply double the size of every coin. It's also true that every representation of $2^k$ pfatz using the first $k$ coins gives rise to a representation of $2^{k+1}$ pfatz using the first $k + 1$ coins by just adding a $2^k$ pfatz piece. The moral is: $w(2^{k+1}, k + 1) = 2 \cdot w(2^k, k)$. This enables us to prove the left-hand inequality by induction on $k$.

To prove the right-hand inequality we note that any manifestation of $2^k$ pfatz using smaller coins can be tho'rt of as a list of length $k$ where the $i$th member of the list tells us how many $2^i$ pfatz coins we are using. How many lists of length $k$ each of whose entries are at most $2^k$ are there? Answer $(2^k)^k$, which is $2^{k^2}$.

## 1997:1:2

## 1997:1:7

(a) Yes: equality is a partial order, and it is tree-like because the set of strict predecessors is always empty.

(b) Yes. The usual order is a partial (indeed *total*) order and every total order is tree-like.

(c) No. This is a partial order but is not tree-like because (for example) 6 has two immediate strict predessors.

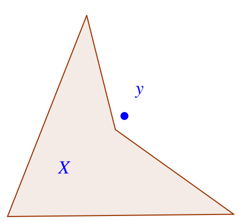
(d) This is reflexive and antisymmetrical (if $x\,R\,y$ and $yRx$—so that $x$ and $y$ are either equal or each is the greatest prime factor of the other—then they are equal). The hard part is to show that it is transitive. Suppose $x\,R\,y$ and $y\,R\,z$. If $x = y$ or $z = z$ we deduce $x\,R\,z$ at once, so consider the case where $x\,R\,y$ and $y\,R\,z$ hold, but *not* in virtue of $x = y$ or $y = z$. But this case cannot arise, because if $y\,R\,z$ and $y \neq z$, then $y$ is a prime, and the only $x$ such that $x\,R\,y$ is $y$ itself. Finally, it's easy to show this relation is tree-like, because no number can have more than one greatest prime factor.

It seem to me that the number of treelike partial orderings of $n$ elements is precisely $n!$. Each treelike partial ordering of $n$ chaps gives rise to $n$ new partial orderings because the extra chap can be stuck on top of any of the $n$ things already there. No new partial ordering gets counted twice.

## 2002:1:8

The last part seems to have caused problems for some. Let's have a look.

We are contemplating relations that hold between elements of $\Omega$ and subsets of $\Omega$. An example of the sort of thing the examiner has in mind is the relation that a point $y$ in the plane bears to a (typically non-convex) region $X$ when $y$ is in the convex hull of $X$.




The idea is that $y$ is one of the points you have to "add" to obtain something convex. (Check that you know what a convex set is, as i'm going to procede on the assumption that you do, and use it as a—one hopes!—illuminating illustration)

What is $\mathcal{R}$? $\mathcal{A}$ is an intersection-closed family of subsets of $\Omega$. (As it might be, the collection of convex subsets of the plane). We are told that it is the relation that relates $y$ to $X$ whenever anything in $A$ that extends $X$ also contains $y$. In our illustration—where $\mathcal{A}$ is the collection of convex subsets of the plane— $\mathcal{R}$ is the relation that hold between $X$ and $y$ whenever $y$ is in the convex hull of $X$. (If you don't already know the meaning of the expression "convex hull" you can probably guess it from the news that, in the picture above, $y$ is in the convex hull of $X$.) Certainly in this case any set that is $\mathcal{R}$-closed is convex.

Assume $C$ is $\mathcal{R}$-closed. That is to say

$$\forall (X, y) \in \mathcal{R}. X \subseteq C \to y \in C \tag{1}$$

(That's in their notation: i'd've written it $(\forall \langle X, y \rangle \in \mathcal{R})(X \subseteq C \to y \in C)$ which (i think) makes the scoping clearer.)

But $\mathcal{R} = \{(X, y) \in \mathcal{P}(\Omega) \times \Omega : (\forall A \in \mathcal{A})(X \subseteq A \to y \in A)\}$. Substituting this for '$\mathcal{R}$' in (1) we obtain

$$\forall (X, y) \in \{(X, y) \in \mathcal{P}(\Omega) \times \Omega : (\forall A \in \mathcal{A})(X \subseteq A \to y \in A)\}. X \subseteq C \to y \in C \tag{2}$$

which reduces to

$$(\forall X, y)[(\forall A \in \mathcal{A})(X \subseteq A \to y \in A) \wedge X \subseteq C. \to y \in C] \tag{3}$$

The examiners suggest you should consider the set $\{A \in \mathcal{A} : C \subseteq A\}$. I think they want you to look at $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$.

If you've followed the action this far you would probably think of this anyway, since this is a set that you know must be in $\mathcal{A}$ and it seems to stand an outside chance of being equal to $C$. So let's look again at (3) to see if it does, in fact, tell us that $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$ is $C$.

And—of course—it does. First we instantiate '$X$' to '$C$' in (3) to obtain:

$$\forall y[(\forall A \in \mathcal{A})(C \subseteq A \to y \in A) \to y \in C] \tag{4}$$

Now let $y$ be an arbitrary member of $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$. That means that $y$ satisfies the antecedent of 4. So it satisifes the consequent of 4 as well. So we have proved that $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$ is a subset of $C$. It was always a superset of $C$, so it is equal to $C$. So $C \in \mathcal{A}$ as desired.