A Question adapted from 1995:5:4X (maths 1a)

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Here is an `ML` declaration of two functions by a simultaneous (or "mutual") recursion.

```
fun f n = if n = 0 then 0 else g(f(n-1) + 1, 1) -1
and g(n,m) = f(f(n-1)) + m + 1;
```

Question: What are the running times of `f` and `g`?

The mutual recursion gives us a pair of mutual recurrence relations:

A: $F(n) = G(\mathtt{f}(n-1)+1, 1) + F(n-1)$
B: $G(n, m) = F(n-1) + F(\mathtt{f}(n-1)) + k$

where F is the cost function for `f` and G is the cost function for `g`. That is to say, $G(n, m)$ is the time taken to compute $\mathtt{g}(n, m)$ and $F(n)$ is the time taken to compute $\mathtt{f}(n)$. Notice that the function `f` appears in the equations A and B, and life would be a bit easier if we could simplify A to get rid of it. So what is `f`? A quick investigation suggests that $(\forall n \in \mathbb{N})(\mathtt{f}(n) = n)$, but we had better prove it! It's true for $n = 0$. For the induction step the recursive declaration tells us that

$\mathtt{f}(n+1) = \mathtt{g}(\mathtt{f}(n)+1, 1) - 1$ (by substituting $n+1$ for n)

But $\mathtt{f}(n) = n$ by induction hypothesis so this becomes

$\mathtt{f}(n+1) = \mathtt{g}(n+1, 1) - 1$

Now, substituting $(n+1)$ for n and 1 for m in the declaration for `g` we get

$\mathtt{g}(n+1, 1) = (n+1-1) + 1 + 1$

which is $n+2$ giving $\mathtt{f}(n+1) = n+1$ as desired. ■

Now that we know $\mathtt{f}(n) = n$ we can use this fact to simplify our recurrence relations as follows.

A′: $F(n) = G(n,1) + F(n-1)$

B′: $G(n,m) = F(n-1) + F(n-1) + k$ whence

B″: $G(n,m) = 2 \cdot F(n-1) + k$

This gives

$$F(n) = F(n-1) + F(n-1) + F(n-1) + k$$

so $F(n) \in O(3^n)$.

G is exponential too. We have assumed that the cost of adding the second agument (‘m’) is constant, but altho’ this simplification will cause no problems it is a simplification nevertheless. Adding two arguments takes time proportional to the logarithm of the larger of the two. Fortunately the cost functions of these algorithms are so huge that an extra log or two will make no difference to the order.