Permutations and the axiom of choice

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1 Introduction

The purpose of this paper is to explore the connections between permutation groups, definability, and the axiom of choice (AC). The fact that these three topics are closely related is perhaps rather well known, but it is possible to bring out into the open some of the precise connections, which may be spelt out in quite a systematic way. To start from a particular point of view, in the original models for the negation of the axiom of choice there was an explicit use of permutation groups. I refer to the Fraenkel–Mostowski method. Moreover, this was developed to a high degree of elegance, particularly in the work of Mostowski where he was able, by careful choice of the group and the 'support structure', to illustrate non-implications between various weak versions of AC. For instance, he found a model in which every set can be *linearly* ordered (the *ordering principle*) but AC fails (Mostowski, 1939), and (1945) he investigated the connections between versions of AC for families of *finite* sets. More sophisticated constructions were used by other authors, for instance Läuchli (1964), who gave a model in which any family of non-empty finite sets has a choice function, but the ordering principle fails.

The Fraenkel–Mostowski method was makeshift in the sense that at the time when it was devised no-one knew how to construct models of Zermelo–Fraenkel set theory in which AC was false. For this reason, a related system in which models could possess non-trivial $\in$-automorphisms was used. After the advent of forcing, most results obtained by FM methods could be transferred straight to ZF. To begin with, these were all done on an *ad hoc* basis, but then more precise methods of direct transfer were discovered. The first general result of this sort was the Jech–Sochor Theorem (Jech and Sochor, 1966), and the method was considerably extended by Pincus (1972), relying principally on a careful analysis of the Halpern–Levy model (Halpern and Levy, 1971).

Since in ZF there can be no non-trivial $\in$-automorphisms of the universe, if we are to use permutational methods the permutations must lie elsewhere. In Cohen's work he viewed these as acting on the 'label space'. In the more streamlined boolean-valued treatment of forcing due to Scott and Solovay (described by Bell (1985)), the permutations are automorphisms of a complete boolean algebra. Here they extend not to automorphisms of the universe, but rather to automorphisms of the *boolean-valued* universe. If the intuition behind the Fraenkel–Mostowski method was that to negate AC we need to find a symmetrical family of sets such that any choice function for it has to be asymmetrical (and hence excluded), the modified intuition for Cohen models is that we should find a family which with boolean value **1** is sufficiently symmetrical, but such that any choice function is not. This cannot be directly because of lack of symmetry in the true ZF universe, but it can be because of lack of symmetry in its boolean counterpart, or perhaps better, because of lack of *definability*. This is where the other theme, of definability, comes into the picture.

In Section 2 we give a recap of the Fraenkel–Mostowski method, illustrating by giving a standard construction required later. Next in Section 3 we examine some classical independence proofs related to weak versions of the axiom of choice, and show how a uniform treatment can be given by using universal-homogeneous structures. This follows ideas of Pincus (1976). A particularly beautiful illustration of the relationship between $\neg$ AC models and permutation groups comes about in the Mostowski–Gauntt theory of finite versions of the axiom of choice, where the permutation groups are finite. We describe the corresponding 'Galois theory' for this situation in Section 4, emphasizing the different ways in which this works out corresponding to implications between the various modified finite versions of the axiom of choice studied. Then in Section 5 we explore a fascinating area where the connections with model theory seem to be very strong, and describe work still in progress (Creed, to appear; Truss, in press) concerning set-theoretic analogues of the model-theoretic notion of 'strongly minimal set', where we aim to perform a 'classification' of so-called *amorphous* and *o-amorphous* sets.

2 Definability, choice, and Fraenkel–Mostowski models

I shall now recall the classical construction of Fraenkel and Mostowski, which is used to form models in which the axiom of choice is false. The method predates Cohen's techniques by many years, and was at that time the best that could be done in obtaining $\neg$ AC models. It was carried out in the context of a modified set theory which can accommodate the existence of 'atoms' (or *urelemente*). These are objects which are not sets, but which can be members of sets. The point of allowing atoms is the following.

The key idea in obtaining a model for $\neg$AC is to obtain a set which is reasonably 'symmetrical', but such that any choice function for its members must be rather 'asymmetrical'. Since atoms are by their very nature set-theoretically indistinguishable, the notions of symmetry and asymmetry of sets built up from them make good sense, and if we have so arranged things that our universe contains only symmetrical objects, a set may lie in the model without there being any choice function for it.

The Fraenkel–Mostowski method takes this approach fairly literally; that is, there really *is* a symmetry group acting, and we do apply a criterion of how symmetrical sets should be for them to lie in the model. The group elements are taken to be automorphisms of the set-theoretical universe. This explains why in this approach it is necessary to assume there are atoms. For, in ordinary ZF set theory, one can prove by transfinite induction on rank that the only $\in$-automorphism of the universe is the identity. This is because if g is an $\in$-automorphism then for every set x, $xg = \{yg : y \in x\}$.

We work therefore in the theory FM, which is obtained from ZF by modifying the axiom of extensionality to allow the existence of atoms. It now says that if two sets are non-empty, and have the same members, then they are equal. The empty sets are then either the 'true' empty set, or atoms. To distinguish which are which we adjoin a unary predicate symbol $U(x)$, to express 'x is an atom'. Jech (1970) shows how to find a model for FM built out of certain of the sets of a model for ZF (which we may think of as adjoining the desired atoms 'at the side'), so there is no problem about the existence of such a thing (relative to that of a model of ZF).

Let $\mathfrak{M}$ therefore be a model of FMC ($=$ FM $+$ AC) in which the family U of atoms forms a set (though there are versions of the method where U is a class). Suppose that G is a group of permutations of U, and that $\mathscr{F}$ is a filter of subgroups of G closed under conjugacy. These are then the ingredients necessary for construction of an FM model. Before we can do this we first have to say how G will act on the members of $\mathfrak{M}$. This is uniquely determined by the requirement that it should respect the membership relation, so as above we have to let $xg = \{yg : y \in x\}$ for each x. Once G has been allowed to act on the whole of $\mathfrak{M}$, we at once obtain stabilizers in the usual way. It is convenient to distinguish setwise and pointwise stabilizers, which are written as $G_{\{x\}} =_{\text{def}} \{g \in G : xg = x\}$ (the setwise stabilizer of x) and $G_{(x)} =_{\text{def}} \{g \in G : yg = y \text{ for all } y \in x\}$ (the pointwise stabilizer of x). The resulting FM model $\mathfrak{N}$ then consists of those members of $\mathfrak{M}$ which are hereditarily symmetric with respect to $\mathscr{F}$. Thus we may write $\mathfrak{N} = \{x \in \mathfrak{M} : x \subseteq \mathfrak{N} \text{ and } G_{\{x\}} \in \mathscr{F}\}$ (which is a definition by transfinite induction).

It is necessary to check that this gives a model of FM. That means that all the axioms of FM are to hold when relativized to $\mathfrak{N}$. Now by definition, $\mathfrak{N}$ is certainly *transitive*, meaning that any member of a member of $\mathfrak{N}$ is itself in $\mathfrak{N}$, and from this extensionality (modified of course) and foundation hold. All the 'standard' sets, those whose transitive closures contain no atoms, are automatically in $\mathfrak{N}$, so the axiom of infinity holds, and the axioms of union and power set are straightforward to check (where the power set of $X \in \mathfrak{N}$ in $\mathfrak{N}$ equals the intersection of the power set of X in $\mathfrak{M}$ with $\mathfrak{N}$). It is for the axiom of replacement that the closure of $\mathscr{F}$ under conjugacy is required. Details are given in Jech (1970, Lemma 100).

The idea behind the failure of the axiom of choice in $\mathfrak{N}$ (for suitable choice of group and filter), is that many sets will have been put into $\mathfrak{N}$ on the grounds that they are 'symmetrical enough' (that is, their stabilizer lies in $\mathscr{F}$), but that any well-ordering for them (or choice function) would have to be asymmetrical, and hence will be absent. Let us give a classical case, for constructing a model containing what in Truss (in press) I call a 'strictly amorphous set', due to Fraenkel (1922) (though it was Mostowski (1938) who actually showed that U is amorphous there).

Here we take U to be infinite, and let G be the group of all permutations of U (in $\mathfrak{M}$). For $\mathscr{F}$ we take the filter generated by the stabilizers of the members of U. More precisely

$$\mathscr{F} = \{H \leqslant G : \exists A \subseteq U\, (A \text{ finite and } H \geqslant G_{(A)})\}.$$

Since $g^{-1}G_{(A)}g = G_{(Ag)}$, it is immediate that $\mathscr{F}$ is closed under conjugacy. This defines the model. The idea is that U will be the strictly amorphous set in $\mathfrak{N}$, so first we should check that U *does* lie in $\mathfrak{N}$. Each member of U lies in $\mathfrak{N}$ since it has no members itself, and its stabilizer was explicitly put into the filter. It follows that U is in $\mathfrak{N}$, now that we have shown that all its members are, since its stabilizer equals G. It is at the *next* step that many sets of $\mathfrak{M}$ have been omitted. In fact the only subsets of U which are in $\mathfrak{N}$ are those which are finite or cofinite (= complement of finite). For suppose that $X \subseteq U$ lies in $\mathfrak{N}$ and is infinite. There must therefore be a finite subset A of U such that $G_{(A)} \leqslant G_{\{X\}}$. Since X is infinite there is $x \in X - A$, and as $G_{(A)}$ acts transitively on $U - A$, X must contain the whole of $U - A$, so is cofinite.

Saying that a set is *amorphous* means that it is infinite, but is not the disjoint union of two infinite sets. What we have therefore seen is that U is amorphous in $\mathfrak{N}$. The existence of an amorphous set clearly contradicts the axiom of choice, so that AC is false in $\mathfrak{N}$. One way of viewing amorphous sets is that they are infinite sets which are 'Dedekind finite' in a very strong sense, and this is how they are viewed for instance in Levy (1958) and Truss (1974*a*). Recall that a set is said to be *Dedekind finite* if it has no countably infinite subset. The model just described, and that of Mostowski (1939), are the classical examples of models containing infinite but Dedekind finite sets.

A stronger statement is also true of U, as is expressed by saying that it is strictly amorphous. This just means in addition that in any partition of the set into infinitely many pieces, all except finitely many of the pieces are singletons. Note that this is analogous to the notion of a *strictly minimal set* in model theory, indicating the first of the many connections between axiom of choice properties and ideas in model theory. If the fact that U is amorphous in $\mathfrak{N}$ corresponds to the transitivity of $G_{(A)}$ on $U - A$, the fact that it is strictly amorphous corresponds rather to its *primitivity* there. The easy proof is omitted.

To analyse the properties of a model of this sort in more detail, one really needs a 'support structure'. This consists of a mechanism for relating properties of arbitrary sets in the model to the behaviour of G and $\mathscr{F}$. It is conceptually easiest to handle when there is some global form of AC available, as is the case in L, (the 'constructible universe'), since then one can assert the existence of a (definable) class which supplies the support for all sets simultaneously. This is not really necessary however, and local choice, that is choice for arbitrary families of non-empty sets (rather than choice for the the class of *all* sets) is quite adequate. The idea of a 'support' is defined as follows. We say that one set, x, *supports* another, y, if $G_{(x)} \leqslant G_{\{y\}}$, that is to say, any member of G which fixes every element of x also fixes y. The key point which makes the reduction work is that any family all of whose members are supported by one fixed set can be well-ordered in the model. This is because (as is easy to see) a set can be well-ordered in an FM model if and only if its *pointwise* stabilizer lies in $\mathscr{F}$.

Now by definition, any set in our model has a support which is a finite subset of U. What we want is to be able to make a simultaneous choice of supports for all sets. Since any set will have many supports (for instance, if A supports x and $B \supseteq A$ then B also supports A), this is not always immediate. If x has a *minimal* support, then things generally work out well, and that is indeed what happens here. The relevant lemma is as follows:

Lemma 2.1. For any finite subsets A and B of U,

$$G_{(A\cap B)} = \langle G_{(A)}, G_{(B)} \rangle.$$

Remark. To see that this is sufficient to obtain minimal supports, let x be any set in $\mathfrak{N}$. Then there is some finite $A \subseteq U$ such that $G_{\{x\}} \geqslant G_{(A)}$. Let such A be chosen of least possible cardinality. Then it is contained in any other support B for x, for as $G_{(A\cap B)} = \langle G_{(A)}, G_{(B)} \rangle$, $A \cap B$ is also a support for x, so by minimality of $|A|$, $A \subseteq B$. The precise use to which minimal supports are put will be illustrated in Section 3.

Proof of Lemma 2.1. That $G_{(A\cap B)} \geqslant \langle G_{(A)}, G_{(B)} \rangle$ is immediate. Conversely, let $g \in G_{(A\cap B)}$. Let $h \in G_{(B)}$ be such that $(A-B)h \cap (A \cup B)g^{-1} =$

$\varnothing$. Then $hg \in G_{(A\cap B)}$ and $(A\cup B)\cap(A-B)hg=\varnothing$. Define k by

$$xk = \begin{cases} xhg & \text{if } x \in A-B, \\ x(hg)^{-1} & \text{if } x \in (A-B)hg, \\ x & \text{otherwise.} \end{cases}$$

Then $k \in G_{(B)}$ and $hgk \in G_{(A)}$, so that

$$g = h^{-1}(hgk)k^{-1} \in \langle G_{(A)}, G_{(B)}\rangle. \qquad \square$$

Since it is part of our thesis that the notions of definability and choice are closely related, let us remark that there is an alternative description of this model. We may say that it consists precisely of those members of $\mathfrak{M}$ which are hereditarily definable over U, with standard sets also allowed as parameters. Here when we talk about definability *over* U, we mean that members of U may be employed as parameters of the definition. And by a 'standard' set we just mean one whose transitive closure contains no atoms. The relevant material about definability was all discussed in Myhill and Scott (1971) (where it was *ordinals* which were allowed as extra parameters, but the same techniques apply here). The facts that all the members of U, and U itself, are then in this model are immediate. Moreover it is rather clear that it is the same model as was previously defined permutationally. The construction is more direct, one could say; for here we explicitly 'put into' the model exactly the sets we want—U, the members of U, and unavoidably (in view of the axioms of set theory) the finite and cofinite subsets of U. But *no other* subsets of U. It has to be said however that the models are generally easier to work with in terms of the permutational definition.

The great advantage of the Fraenkel–Mostowski method is its simplicity. In a ZF framework one has to work quite a lot harder to achieve the same effect. Either a permutational approach, or one using ideas of definability is possible. Even in the latter case, many of the arguments work out best using permutations—as one can see by looking at some of the key papers (Cohen, 1966; Levy, 1966; Solovay, 1970). In a Fraenkel–Mostowski model one can concentrate on the issues concerned with the axiom of choice which are genuinely features required in the construction, avoiding additional complications needed to make the forcing work. And one can argue, as is done in Truss (in press), that for certain questions, FM is in any case the 'right' theory to be working in when focussing on the model-theoretic rather than the set-theoretic aspects.

3 Classical independence proofs

Levy (1965) summarized several of the independence questions which at that time were still unresolved between weak versions of the axiom of choice. Many of these have since been settled.

The main ones considered are as follows:

C_n: The axiom of choice for families of n-element sets ($1 \leqslant n < \aleph_0$), which says that any family of n-element sets has a choice function.

$C_{<\omega}$: The axiom of choice for families of non-empty finite sets.

OP: The ordering principle: any set can be linearly ordered.

OE: The order-extension principle: any partial ordering can be extended to a linear ordering.

BPIT: The boolean prime ideal theorem: any boolean algebra has a prime ideal.

The principal implications between these are as follows:

$$\mathrm{AC} \Rightarrow \mathrm{BPIT} \Rightarrow \mathrm{OE} \Rightarrow \mathrm{OP} \Rightarrow \mathrm{C}_{<\omega}. \qquad (\star)$$

None of the reverse implications is provable (in ZF) as was shown by Halpern, Felgner, Mathias, and Läuchli, respectively. The proofs used involved a variety of techniques, but an elegant idea of Pincus's means that a quite uniform treatment is possible, involving the model-theoretic idea of 'universal-homogeneous structure'. As illustration, let us consider the question as to whether the axiom of choice for families of finite sets implies the ordering principle. This was settled by Läuchli (1964) using a Fraenkel–Mostowski model. In his key lemma he showed that there is a group G having the following properties:

(i) if $A \subseteq \Omega$ is finite there are $g, h \in G_{(A)}$ and distinct $a, b, c \in \Omega$ such that $ag = b$, $bg = c$, and $bh = c$, $ch = a$;

(ii) if $A \subseteq \Omega$ is finite then $G_{\{A\}} = G_{(A)}$ (pointwise stabilizer = setwise stabilizer);

(iii) if $A, B \subseteq \Omega$ are finite then $G_{(A \cap B)} = \langle G_{(A)}, G_{(B)} \rangle$.

His proof was quite involved, and an alternative method, using a combination of forcing and the Fraenkel–Mostowski construction was given in Truss (1974a), using an idea of Gauntt's (Gauntt, 1970). Then in Pincus (1976) it was shown how to do the same thing in a purely FM setting by the use of universal-homogeneous structures. Since this also settles a question raised by P. M. Neumann at one of the Oxford–Queen Mary College series of seminars, let me describe the idea. Neumann's question was the following: note that if G is a permutation group on the set Ω, and G preserves a linear ordering on Ω, then no member of G has a non-trivial finite cycle. Is the converse true? That is to say, if no member of G has a non-trivial finite cycle, is there necessarily a linear ordering on Ω preserved by G? Negative answers to this were rapidly supplied by Cameron and others. In fact Läuchli's group will suffice, (ii) precisely saying that no member of G has non-trivial finite cycles, and (i) telling us that G cannot preserve a linear ordering. The third condition is not needed at all for this particular result.

I shall now present a simplified proof of Läuchli's result based on Pincus's idea, and then describe constructions using various other universal-homogeneous structures (without giving full details), some of which settle the non-implications mentioned in ($\star$).

THEOREM 3.1. There is a permutation group G on a countably infinite set Ω which preserves no linear ordering on Ω and such that no member of G has a non-trivial finite cycle.

Proof. (Pincus) Consider structures of the form $\langle \Omega : f_1, f_2, f_3, \ldots \rangle$, where for each $n \geq 1$, f_n is a choice function for the set $[\Omega]^n$ of n-element subsets of Ω. Moreover suppose that $|\Omega| = \aleph_0$ and that $\langle \Omega : f_1, f_2, f_3, \ldots \rangle$ is universal-homogeneous in the usual sense, that is, it is obtained in a generic fashion as a countable union of finite substructures. Note that strictly speaking these structures are not first-order, but they can easily be replaced by equivalent first-order structures as in Pincus (1976) by instead using for each n an n-ary function f_n which chooses one of its arguments, and is symmetric under interchange of arguments. Let $G = \mathrm{Aut}\,\langle \Omega : f_1, f_2, f_3, \ldots \rangle$. Then it is immediate that no member of G can have a finite non-trivial cycle, since if A is a finite orbit of $g \in G$ with $|A| > 1$, then g cannot preserve $f_{|A|}$ on A.

Now let us check that G satisfies Läuchli's property (i) above, and hence that it can preserve no linear ordering on Ω. Let A be the given finite subset of Ω, and choose distinct a, b, and c not in A. Let $\Omega_1 = A \cup \{a, b, c\}$ and turn Ω_1 into a structure of the same type by defining f'_n for $n \leqslant |\Omega| + 3$ by $f'_{|X|}(X) = f_{|A \cap X|}(A \cap X)$ if $A \cap X \neq \varnothing$, $f'_2\{a, b\} = a$, $f'_2\{b, c\} = b$, $f'_2\{c, a\} = c$, and $f'_3\{a, b, c\} = a$. Note that as we have defined it, $\langle \Omega_1, f'_1, f'_2, f'_3, \ldots \rangle$ need no longer be a substructure of $\langle \Omega, f_1, f_2, f_3, \ldots \rangle$. But as $\langle \Omega, f_1, f_2, f_3, \ldots \rangle$ was chosen to be universal-homogeneous, by changing the choice of a, b, and c, we may suppose that it *is*. It remains to find appropriate g and h. If we define g_1 and h_1 to fix A pointwise, and so that $ag_1 = b, bg_1 = c \,\&\, bh_1 = c, ch_1 = a$, then g_1 is an isomorphism from $A \cup \{a, b\}$ to $A \cup \{b, c\}$, so by homogeneity extends to the required automorphism g of Ω, and similarly for h. □

Next we discuss the other non-implications mentioned above. In each case the construction involves consideration of a suitable universal-homogeneous structure naturally tailored to the problem in hand. This may be defined by Fraïssé's method using a suitable amalgamation class of structures. In addition there has to be an analogue of Mostowski's 'support lemma', which, rather than appealing to the condition given in Pincus (1976), one can verify separately in each case without too much difficulty. Felgner's proof (Felgner, 1972) of OE $\not\Rightarrow$ BPIT used a Cohen model, but

(iv) describes what may be viewed as the 'natural' corresponding universal-homogeneous structure for providing an FM proof. Most of the proof that this construction works is complete, but it remains to verify all cases of OE in the model. In addition we mention two other universal-homogeneous structures needed for consistencies in Section 5.

(i) U has the trivial structure. Here the automorphism group is just $Sym(U)$.
(ii) U carries the structure of a countable-dimensional projective geometry over a finite field, F_q.
(iii) $U \cong (\mathbb{Q}, <)$. The automorphism group consists of the order-preserving permutations.
(iv) U is a countable atomless boolean algebra with a linear ordering extending the partial ordering of the algebra.
(v) U is a countable universal partial ordering $(U, \leqslant)$ with an (independent) linear ordering $\preceq$.
(vi) U is as in the proof of Theorem 3.1.

As mentioned before, a key point in using these structures to obtain the desired consistency is an appropriate support lemma, which enables us to assign minimal supports to all members of the model, and hence to relate the structure of general sets to that of the family of finite subsets of U.

LEMMA 3.2. For each of the permutation groups just listed, and for any finite subsets A and B of U (except that for (ii), A and B have to be subspaces too, and for (iv) they have to be subalgebras),

$$G_{(A \cap B)} = \langle G_{(A)}, G_{(B)} \rangle.$$

Proof. The result was proved for the first case in Lemma 2.1, and the proof of (iii) is essentially the same. For the other cases some modification is required. For instance for (ii) we take $h \in G_{(B)}$ to be such that $(A-B)h$ is disjoint from the *span* of $(A \cup B)g^{-1}$, and k to be some member of G such that $xk = xhg$ if $x \in A - B$, $xk = x(hg)^{-1}$ if $x \in (A-B)hg$, and $xk = x$ if $x \in B$. But we cannot any longer insist that it is the identity for other points (since it must preserve the geometry). Cases (iv), (v), and (vi) are discussed in Felgner and Truss (to appear), Jech (1973), and Pincus (1976) respectively. □

Now let us see how the models resulting from these structures are used to establish the stated FM consistencies. In each case we suppose that $\mathfrak{M}$ is a model of FMC in which U is the set of all atoms, and U is countable. Since the structures described above all have countable domains, we may suppose that U is indexed by any one of them, and we get a natural action of the automorphism group of the structure on U. Let us denote this by

G. The supports are taken to be finite, that is $\mathscr{F}$ is the filter of subgroups of G generated by $\{G_x : x \in U\}$. Let $\mathfrak{N}$ be the resulting FM model.

THEOREM 3.3. The Fraenkel–Mostowski models defined naturally from the six structures listed have the following properties:

(i) U is strictly amorphous;

(ii) U is amorphous but not strictly amorphous, and carries a non-degenerate modular geometry, (we say that it has *projective type*);

(iii) BPIT holds but AC does not;

(iv) BPIT fails;

(v) OP holds but OE does not;

(vi) $C_{<\omega}$ holds but OP does not.

Proof. (i) has already been shown in Section 2, and that U is amorphous in (ii) follows from the fact that the pointwise stabilizer of any finite subset of U acts transitively on a cofinite subset. The geometry on U is preserved by definition of G. Note that we cannot now however index it by the one-dimensional subspaces of an $\aleph_0$-dimensional vector space over F_q. The indexing has been 'lost' in passing from $\mathfrak{M}$ to $\mathfrak{N}$. But the non-degeneracy and modularity have not.

(iii) The proof that BPIT holds in $\mathfrak{N}$ is beyond the scope of this brief survey and involves a certain amount of combinatorics (Halpern and Levy, 1971; Jech, 1973), though an alternative proof, going by way of the compactness theorem and using ideas of Ehrenfeucht and Mostowski, was given by Pincus (1976). That AC is false in $\mathfrak{N}$ is however easy to see, since any subset of U must be a finite union of intervals and points, having finite support A (and as G acts transitively on the open intervals defined by A). This is the prototype of an *o-amorphous* set (see Section 5), and we deduce easily that U has no countable subset, and in particular, cannot be well-ordered.

(iv) Since U is indexed by a countable boolean algebra (with linear ordering) and G preserves the boolean algebra structure, it is still a boolean algebra in $\mathfrak{N}$. To show that BPIT fails in $\mathfrak{N}$ the algebra U itself is used, and it is shown to have no prime ideal in $\mathfrak{N}$. We remark that the construction is specifically designed so that OE should hold in $\mathfrak{N}$. With this object, an extension of the natural partial ordering (as a boolean algebra) of U was explicitly put into $\mathfrak{N}$. It is anticipated that it will follow from this, and the existence of minimal finite supports, that *all* partial orderings in $\mathfrak{N}$ can be extended to linear orderings.

(v) The fact that any set has a unique finite support in the model means that any set can be put into 1–1 correspondence with a subset of $\alpha \times e(U)$ for some ordinal α, where $e(U)$ is the family of finite subsets of U. But the linear ordering $\preceq$ of U can be lifted to a linear ordering of $\alpha \times e(U)$, thus verifying OP in $\mathfrak{N}$. To see that OE fails, we show that the partial ordering $\leqslant$ on U (which in $\mathfrak{M}$ is the countable universal partial ordering) has no extension to a linear ordering in $\mathfrak{N}$. For if $\leqslant^*$ were such an extension, it would have to have finite support A, say. By universality and homogeneity there are x, y, z, t exceeding all members of A in both $\leqslant$ and $\preceq$ such that the only relations between them are given by $x \leqslant y$, $z \leqslant t$, $x \preceq t \preceq y \preceq z$. By looking at the isomorphic substructures $A \cup \{x, z\}$, $A \cup \{x, t\}$, $A \cup \{y, z\}$, $A \cup \{t, y\}$, and using homogeneity again, and the fact that $G_{(A)} \leqslant G_{\{\leqslant^*\}}$, we find that either all or none of $x \leqslant^* z$, $x \leqslant^* t$, $y \leqslant^* z$, $t \leqslant^* y$ hold. Since $x \leqslant y$ we cannot have both $t \leqslant^* x$ and $y \leqslant^* t$. Since $z \leqslant t$ we cannot have both $t \leqslant^* y$ and $y \leqslant^* z$. Thus we have the desired contradiction.

(vi) That any family of finite sets has a choice function in $\mathfrak{N}$ follows by using finite supports, and using the choice functions f_n for the set of n-element subsets of U which have been explicitly included. To see that U cannot be ordered in the model, suppose on the contrary that $\leqslant$ is a linear ordering of U supported by $A \subseteq U$. Then by property (i) we know that there are $g, h \in G_{(A)}$ and distinct $a, b, c \in \Omega$ such that $ag = b$, $bg = c$, and $bh = c$, $ch = a$. But as $G_{(A)}$ preserves $\leqslant$, $a \leqslant b \Leftrightarrow b \leqslant c \Leftrightarrow c \leqslant a$, and as $\leqslant$ is meant to be a linear ordering, this gives a contradiction. □

Let us remark on the transfer of these results to ZF. In the early days of forcing, these transfers were done separately for each model. Then in the Jech–Sochor Theorem (Jech and Sochor, 1966) a method for automatically deriving a ZF consistency corresponding to an FM consistency was given, for statements of a certain special form ('boundable' statements). This was subsequently extended by Pincus (1972). The basic idea is that the 'totally indistinguishable' atoms should be replaced by some 'sufficiently indistinguishable' sets. These may be reals, sets of reals, sets of sets of reals,..., depending on the statement to be transferred. For instance, since an amorphous set cannot be linearly ordered, it is hopeless to try to transfer the consistency of the existence of an amorphous set by replacing the atoms by reals, since any set of reals *can* be ordered. The next thing to try is to represent them as a set of sets of reals, and this turns out to be good enough, in each of (i) and (ii).

For (iii), (iv), and (v), it is actually *simpler* in some respects for the ZF case than for FM. The point is here that there is a ('generic') linear ordering on the structure, and if we take the atoms to be represented as reals, it may be taken as the usual ordering. In the case of Mostowski's ordered model (iii), this is all that is needed. In the next two instances we have to put on additional structure, which for (iv) is a boolean algebra structure whose partial ordering relation restricts that on $\mathbb{R}$ (but is otherwise 'generic'), and for (v) is a generic partial ordering of $\mathbb{R}$ (meaning one which is independent of the usual ordering).

As remarked by Mathias (1974) there are a number of non-transferable FM consistencies. These seem mainly to be based on the statement 'the power set of any well-ordered set can be well-ordered', which implies AC in ZF, but not in FM (Rubin and Rubin, 1976). I conclude this section by considering one such statement. The point of the example is not so much that the statement is non-transferable, but that in the FM case, its truth in a model is guaranteed by a simple group-theoretical condition, and there is a corresponding condition in the ZF case which instead guarantees the truth of a slightly weaker statement. Perhaps rather than saying it is non-transferable, we should say that its naturally transferred version is an appropriate weakening of the original. The first statement is as follows.

(∗) For any set X, if the set $[X]^2$ of 2-element subsets of X has a choice function, then X can be well-ordered.

Now this particular statement implies AC in ZF, but not in FM. To derive AC from it in ZF, we may prove from it that the power set of any well-orderable set can be well-ordered, and then appeal to Rubin and Rubin (1976, p. 76) to derive AC. The weaker statement which we claim corresponds to (∗) in the ZF setting is this:

(∗∗) For any set X, if $[X]^2$ has a choice function, then X can be mapped 1–1 into a set of sets of ordinals.

THEOREM 3.4. In a Fraenkel–Mostowski model $\mathfrak{N}$ defined by U, G, and $\mathscr{F}$, if $\mathscr{F}$ contains a dense set of groups which are generated by involutions, then (∗) holds.

Proof. Suppose the given statement is true, and that F is a choice function in $\mathfrak{N}$ for $[X]^2$. Let $G_{\{X\}} \cap G_{\{F\}} \geqslant H \in \mathscr{F}$, where H is generated by involutions. We show that $G_{(X)} \geqslant H$. If not, let $g \in H - G_{(X)}$. Write g as a product of involutions. At least one of these must be in $H - G_{(X)}$, so we may assume that $g^2 = 1$. Let $x \in X$, $xg \neq x$. Thus $F\{x, xg\} = y \in \{x, xg\}$, so $(\{x, xg\}, y) \in F$. As $g \in H \leqslant G_{\{F\}}$, $(\{xg, xg^2\}, yg) \in F$. As $xg^2 = x$, $yg = y$, a contradiction. Therefore X is pointwise fixed by H, and we deduce that X can be well-ordered in $\mathfrak{N}$.

As examples of models where this occurs we may take $|U| = \kappa$ for any infinite cardinal κ, $G = Sym(U)$, and let $\mathscr{F}$ be generated by

$$\{G_{(A)} : A \subseteq U, |A| < \kappa\}.$$

This generalizes Fraenkel's model above (which was the case $\kappa = \aleph_0$). □

The analogous result in ZF is obtained by regarding the relevant models of ZF as also being formed by passing to a symmetric submodel. That is, if we work in the boolean-valued universe $V^{\mathbf{B}}$, where $\mathbf{B} \in V$ is a complete boolean algebra, we may suppose that G is a group of automorphisms of

B, and that $\mathscr{F}$ is a filter of subgroups of G closed under conjugacy. The resulting symmetric boolean extension of V consists of all boolean terms which are hereditarily symmetric with respect to $\mathscr{F}$, and the corresponding ZF model is obtained as usual by choice of a generic ultrafilter on **B**. See Jech (1973) for more details.

THEOREM 3.5. In a Cohen model defined by **B**, G, and $\mathscr{F}$, if $\mathscr{F}$ contains a dense set of groups which are generated by involutions, then $(**)$ holds.

The proof, which we omit, is similar to that of the previous theorem. See Truss (1978) for instance.

In this section I have tried to illustrate how certain consistencies can be achieved by examining appropriate universal-homogeneous structures, and building Fraenkel–Mostowski models based on them as in Pincus (1976). Another possible question is to start with the universal-homogeneous structure, and to ask what the properties of the resulting model then turn out to be. Two examples which might be worth investigating are the random graph, and the countable atomless boolean algebra. One could ask then for a closer tie-up between the properties of the structure one started with, and those of the model.

4 Finite versions of the axiom of choice

A beautiful illustration of the relationship between choice and symmetry arises in the consideration of the finite versions of the axiom of choice studied initially by Tarski and Mostowski, and later by others. Here, since it is provable in ZF that any finite family of non-empty sets has a choice function, by a 'finite' version of the axiom of choice we must mean that there is a choice function for a *family* of finite sets. Mostowski realized that in trying to choose 'effectively' a member of a finite set (or any set, come to that), what is essentially involved is achieving a reduction in symmetry. Before the choice is made, there is no particular reason to prefer one member of the set to another, so they all have an equal status, but in making a selection, some reason has to be found for preferring one element to another. As a result of this observation, he formulated various combinatorial conditions on appropriate finite permutation groups which characterize implications between different finite versions of the axiom of choice, very much in the spirit of Galois theory. I briefly recall the main idea of elementary Galois theory, to illustrate the analogies with the present question.

Consider the solution of algebraic equations where 'radicals' are permitted. This means that, as well as performing standard algebraic manipulations such as addition and multiplication, we are allowed to extract roots—square roots, cube roots, and so on. The key observation made by Galois, which made his analysis possible, was that the extraction of a root precisely corresponds to a reduction of symmetry among the roots. To begin with, there is nothing to choose between the three roots α, β, and γ, of a cubic equation, shall we say, and this is more formally expressed by saying that the 'data' consisting as it does of the elementary symmetric functions $\alpha+\beta+\gamma$, $\alpha\beta+\alpha\gamma+\beta\gamma$, $\alpha\beta\gamma$ is unchanged under the action of all permutations of $\{\alpha,\beta,\gamma\}$. Now any other symmetric (rational) function of $\{\alpha,\beta,\gamma\}$ can be expressed in terms of the three particular symmetric functions so given, and the key to solving the equation is finding a (more) asymmetric function whose *square* is symmetric; here $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ will do; and then a totally asymmetrical function whose cube has the same symmetry as $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$; here $\alpha+\beta\omega+\gamma\omega^2$ will do, where ω is a primitive cube root of unity. This computation corresponds to the choice of subgroups

$$S_3 \rhd A_3 \rhd 1$$

and the type of radical extracted to the index of the relevant subgroup in the next larger one.

We now introduce the principal variants of the finite axiom of choice we study:

C_n: Any family of n-element sets has a choice function.

$\mathrm{C}_n^{\mathrm{o}}$: Any linearly ordered family of n-element sets has a choice function.

C_n^*: Any well-ordered family of n-element sets has a choice function.

C_n^ω: Any countable family of n-element sets has a choice function.

Let us first make some easy remarks about these.

LEMMA 4.1. C_0 is false; C_1 is true; $\mathrm{C}_n \Rightarrow \mathrm{C}_n^{\mathrm{o}} \Rightarrow \mathrm{C}_n^* \Rightarrow \mathrm{C}_n^\omega$; if k divides n then $\mathrm{C}_n \Rightarrow \mathrm{C}_k$, and similarly for $\mathrm{C}_n^{\mathrm{o}}$, C_n^*, and C_n^ω.

Proof. (Last part) We let $mk=n$, and suppose that X is a family of k-element sets and Y is a fixed m-element set. Then $\{x\times Y : x\in X\}$ is a family of n-element sets. By C_n it has a choice function f. Define g by $g(x)=\xi$ if $\exists\eta\, g(x{\times}Y)=(\xi,\eta)$. Then g is a choice function for X. □

This proof is due to Tarski, who derived various basic results about implications between the C_n, which were greatly extended by Mostowski (1945). Finally Gauntt (1970) proved that one of Mostowski's conditions, $D(n,Z)$ (see below), is necessary and sufficient for $\forall m{\in}Z\, \mathrm{C}_m \to \mathrm{C}_n$ to be provable. Following a suggestion of A. Levy, I extended these results in various directions (Truss, 1973) (and he independently undertook a generalization, following a slightly different approach (Levy, 1973)). For instance an analysis of the $\mathrm{C}_n^{\mathrm{o}}$, and C_n^*, and the connections between them, was given, and 'mixed' kinds were allowed. Also Z could be infinite, as I now discuss.

If Z is infinite, the natural composite finite choice axiom to take is $\forall n{\in}Z\, \mathrm{C}_n$. But this is weaker than the principle we actually wish to consider, which is

C_Z: If X is a family of sets such that $\forall x{\in}X\ |x| \in Z$, then X has a choice function

(similarly for $C^{\circ}_Z, C^{*}_Z, C^{\omega}_Z$). That $C_Z \Rightarrow \forall n{\in}Z\ C_n$ is clear, but the converse is false for infinite Z, since the existence of separate choice functions f_n for $\{x \in X : |x| = n\}$ for each $n \in Z$ does not at all mean that a simultaneous choice for all $n \in Z$ is possible.

So much for the finite versions of the axiom of choice considered. We next introduce the group-theoretical conditions used by Mostowski (1945) and Gauntt (1970) to analyse the interconnections between them.

$D(n, Z)$: For any fixed-point-free subgroup G of S_n there are a subgroup H of G and proper subgroups $K_1, \ldots, K_r$ of H such that $\sum |H : K_i| \in Z$.
$L(n, Z)$: For any fixed-point-free subgroup G of S_n there are proper subgroups $K_1, \ldots, K_r$ of H such that $\sum |G : K_i| \in Z$.
$K(n, Z)$: For any fixed-point-free subgroup G of S_n there are a subgroup H of G^m, some m, and proper subgroups $K_1, \ldots, K_r$ of H such that $\sum |H : K_i| \in Z$.
$M(n, Z)$: If $n = p_1 + \cdots + p_s$ is an expression for n as the sum of (not necessarily distinct) primes, then there are $\alpha_i \geqslant 0$ such that $\sum \alpha_i p_i \in Z$.

In addition, in Truss (1973) I considered versions of these where S_n is allowed to act on the structure built up from an n-element set by taking power sets ω times. The notation is as follows. Let $e(X)$ be the set of finite subsets of the set X, and $e_n(X)$ be given inductively by $e_0(X) = X$, $e_{n+1}(X) = e(e_n(X))$, $e_\omega(X) = \bigcup_{n\in\omega} e_n(X)$. We choose some fixed n-element set X_n such that each member of $e_\omega(X_n)$ 'appears' only once. For instance X_n may consist of any n distinct infinite subsets of ω. There are three further conditions which we wish to consider.

$A(n, Z)$: For any fixed-point-free subgroup G of $\mathrm{Sym}(X_n)$ there is Y in $e_\omega(X_n)$ such that $|Y| \in Z$ and $\forall \eta{\in}Y\ G_{\{Y\}} \cap G \not\leqslant G_{\{\eta\}}$.
$B(n, Z)$: For any fixed-point-free subgroup G of $\mathrm{Sym}(X_n)$ there is Y in $e_\omega(X_n)$ such that $|Y| \in Z$, $G_{\{Y\}} = G$ and $\forall \eta{\in}Y\ G \not\leqslant G_{\{\eta\}}$.
$C(n, Z)$: For any fixed-point-free subgroup G of $\mathrm{Sym}(X_n)$ there is Y in $e_\omega(X_{n\cdot m})$ for some m such that $|Y| \in Z$ and $\forall \eta{\in}Y\ G^m \not\leqslant G^m_{\{\eta\}}$, where G^m acts on $X_{n\cdot m}$ in the natural way.

The main results about these, obtained by combining the results of Mostowski (1945), Gauntt (1970), Truss (1973) are as follows.

THEOREM 4.2. The following are true:

(i) $D(n, Z) \Leftrightarrow A(n, Z) \Leftrightarrow (C_Z \to C_n) \Leftrightarrow (C_Z \to C^{\circ}_n)$;
(ii) $L(n, Z) \Leftrightarrow B(n, Z) \Leftrightarrow (C^{\circ}_Z \to C^{\circ}_n) \Leftrightarrow (C^{\circ}_Z \to C^{*}_n) \Leftrightarrow$
$(C^{\circ}_Z \to C^{\omega}_n) \Leftrightarrow (C^{*}_Z \to C^{*}_n) \Leftrightarrow (C^{*}_Z \to C^{\omega}_n) \Leftrightarrow (C^{\omega}_Z \to C^{\omega}_n)$;

(iii) $M(n,Z) \Leftrightarrow K(n,Z) \Leftrightarrow C(n,Z) \Leftrightarrow (\mathrm{C}_Z \to \mathrm{C}^*_n) \Leftrightarrow$
$(\mathrm{C}_Z \to \mathrm{C}^\omega_n)$;

(iv) $\mathrm{C}^\circ_Z \not\to \mathrm{C}_n$, $\quad \mathrm{C}^*_Z \not\to \mathrm{C}^\circ_n$, $\quad \mathrm{C}^\omega_Z \not\to \mathrm{C}^*_n$;

where by an implication such as $\mathrm{C}_Z \to \mathrm{C}^\circ_n$ we mean 'is provable in FM' (or ZF actually).

The idea behind the proofs is to show that the effective choice of an element from a set X corresponds to the selection of less and less symmetrical 'objects'. These may be elements of the set, subsets, sets of sequences of elements of the set, and so on, in short, members of $e_\omega(X)$. Rather than giving details of this, I shall illustrate how it works out in one or two cases, parallelling the Galois theory example given above. Observe first that $L(4,\{2,3\})$ is true, which means that $\mathrm{C}^*_{\{2,3\}} \to \mathrm{C}^*_4$ is provable. (On the other hand, $L(n,\{2,3,\ldots,n-1\})$ is false for $n \geqslant 5$. The alternating group A_n provides a counter-example, as it has no subgroup of index less than n). I tabulate below an appropriate chain of subgroups and corresponding choices in $e_\omega(X_4)$.

S_4	$\{a,b,c,d\}$ $\quad \{\{\{a,b\},\{c,d\}\},\{\{a,c\},\{b,d\}\},\{\{a,d\},\{b,c\}\}\}$
$\vert$	applying C^*_3 to the set of 3 2+2 partitions
$C_2 \,\mathrm{Wr}\, C_2$	$\{\{a,b\},\{c,d\}\}$
$\vert$	applying C^*_2
$C_2 \times C_2$	$\{a,b\}$
$\vert$	applying C^*_2 again
$\{1\} \times C_2$	a

The key difference here between $D(n,Z)$ and $L(n,Z)$ is that at each stage in applying $L(n,Z)$ ($\Leftrightarrow B(n,Z)$) to a set derived from a well-ordered family of n-element sets, there is only one element corresponding to each member of the family, so that it is still well-ordered. If only the weaker condition $D(n,Z)$ holds ($\Leftrightarrow A(n,Z)$) we have to be satisfied with *several* sets corresponding to each original set, so that the family to which finite choice is to be applied need no longer be well-orderable. Thus the example just given can be modified to show that $\mathrm{C}_2 \to \mathrm{C}_4$ by choosing a member from each pair of doubletons on the first line, and then one from each doubleton, after which there is sufficient asymmetry to select a single member of $\{a,b,c,d\}$.

The reason Mostowski failed to prove that $\mathrm{C}_Z \to \mathrm{C}_n \Rightarrow D(n,Z)$ was that he concentrated on the case of X well-ordered, and so the best he could hope for was $\mathrm{C}_Z \to \mathrm{C}_n \Rightarrow K(n,Z)$. The first value of n for which $K(n,Z)\ \&\ \neg D(n,Z)$ can hold is 15, where $Z=\{3,5,13\}$. Observe that

$K(n,Z) \Leftrightarrow M(n,Z)$, which holds here:

$$\begin{array}{rclcl} 15 & = & 3+3+3+3+3 & = & 5+5+5 & = & 7+5+3 \\ & & 3.1=3 & & 5.1=5 & & 5.1=5 \\ & = & 7+2+2+2+2 & = & 11+2+2 & = & 13+2 \\ & & 7+3.2=13 & & 11+2=13 & & 13.1=13. \end{array}$$

We now give Gauntt's verification of $\neg D(15,\{3,5,13\})$ (Gauntt, 1970). Let

$$\begin{aligned} a &= (1\,2\,3\,4\,5\,6\,7)(8\,9\,10\,11\,12\,13\,14), \\ b &= (8\,10)(9\,12)(11\,15)(13\,14), \\ c &= (9\,11)(10\,13)(12\,15)(14\,8), \quad \text{and} \\ d &= (10\,12)(11\,14)(13\,15)(8\,9). \end{aligned}$$

Let G be generated by a, b, c, and d and let K be generated by b, c and d. Since b, c, and d commute pairwise, K is abelian of order 8. Now

$$a^{-1}ba = c, \quad a^{-1}ca = d, \quad a^{-1}da = (11\,13)(12\,8)(14\,15)(9\,10) = bd.$$

Hence a normalizes K. Therefore $G = \langle a \rangle K$, so $|G| = 56$. Moreover G is clearly a fixed-point-free group of permutations of $\{1,\ldots,15\}$, and $b^{-1}ab = bab = acb \notin \langle a \rangle$. Hence G has at least 7 elements of order 7. By Sylow's Theorem, G has 48 elements of order 7. If H is a proper subgroup of G whose order is divisible by 7, then again by Sylow's Theorem, H has only 6 elements of order 7. Hence G has no subgroups of order 14 or 28. It follows easily that $\sum |H{:}K_i| \neq 3,5$, or 13.

Despite the failure of $D(15,\{3,5,13\})$, $M(n,Z)$ is known to be sufficient for $\mathrm{C}_Z \to \mathrm{C}_n$ if $n \leqslant 20$ (except $n = 15$), or if n is prime, or if $n = 22, 24, 25, 26, 30, 33, 34, 36, 42, 44, 45$, so the delay in establishing the correct equivalence here is understandable.

Now I consider some slightly different versions of AC, namely of the so-called *axiom of dependent choices*, DC. In this case we make a sequence of choices, each of which may depend on previous ones. The way this works out when the sets are to be finite is as follows.

DC_Z: If T is a finite-branching tree in which every node has exactly n immediate successors for some $n \in Z$, then T has an infinite branch.

Note that in any finitely branching tree, every level is finite, so the axiom of choice for countable families of finite sets implies DC_Z. We may ask however whether, for example, $\mathrm{C}_2^\omega \to \mathrm{DC}_2$ ($\mathrm{C}_2 \to \mathrm{DC}_2$ is clear, by the way). In fact $\mathrm{C}_2^\omega \nrightarrow \mathrm{DC}_2$, and we even have

$$\forall n\, \mathrm{C}_n^* \nrightarrow \mathrm{DC}_m \quad (m > 1).$$

Notice that DC_Z is *not* the same as $\forall n \in Z\, \mathrm{DC}_Z$, even for finite Z. For example, $\mathrm{DC}_{\{2,3\}}$ is not the same as $\mathrm{DC}_2 \,\&\, \mathrm{DC}_3$. This is because $\mathrm{DC}_{\{2,3\}}$ applies to trees where the branching may vary between 2 and 3, whereas DC_2 and DC_3 can only be applied to trees with constant branching degrees. Extending Theorem 4.2 in this case we have

THEOREM 4.3. With the notation of Theorem 4.2, we have

(i) $D(n, Z) \Leftrightarrow (\mathrm{C}_Z \to \mathrm{DC}_n)$,

(ii) $L(n, Z) \Leftrightarrow (\mathrm{DC}_Z \to \mathrm{DC}_n) \Leftrightarrow (\mathrm{DC}_Z \to \mathrm{C}^{\omega}_n) \Leftrightarrow$
$\forall m \in Z\, (\mathrm{DC}_m \to \mathrm{C}^{\omega}_n)$,

(iii) $\mathrm{DC}_Z \not\to \mathrm{C}^*_n, \quad \mathrm{C}^{\circ}_Z \not\to \mathrm{DC}_n$,

and, in addition, the following conjecture:

CONJECTURE 4.4.

$$\forall m \in Z\, (\mathrm{DC}_m \to \mathrm{DC}_n) \quad \Leftrightarrow \quad \text{for some } m \in Z,\ (\mathrm{DC}_m \to \mathrm{DC}_n).$$

I now sketch the essentials of the construction of a Fraenkel–Mostowski model in which C^*_2 holds but DC_2 fails (Truss, 1976). We take for set U of atoms an infinite binary tree, and let G be the group of permutations of U generated by the action of the cyclic group on each set of immediate successors (so that it is a direct limit of finitely iterated wreath products).

If for any group H, we let $\Phi_2(H)$ be the intersection of all subgroups of index 2, then the key point in the construction is to show that there is a proper filter $\mathscr{F}$ of subgroups of G closed under this 'Frattini-like' operation, which may then be used to define an FM model for the desired consistency. In fact, if we let G_n be the subgroup of G consisting of elements which fix pointwise the first n levels of the tree, and act on each subtree starting from the nth level in the same way, then clearly $G \cong G_n$, and we can show that $G_{n+1} \leqslant \Phi_2(G_n)$, for all n, so that $\{G_n : n \in \omega\}$ generates a proper filter $\mathscr{F}$. In the resulting model, U itself then provides a counterexample to DC_2, and the point of the Frattini-like construction of the filter is that this is sufficient to guarantee the truth of C^*_2.

As in the usual Frattini theory, every commutator and every square lies in $\Phi_2(G)$. So we just have to show that every member of G_1 is a product of commutators and squares. Now G_1 is the set of all elements of the form $\langle x, x\rangle$, and G is generated by $G \times G$ (where here the action of the two Gs is taken on the left and right subtrees) and y, where $y^{-1}\langle a, b\rangle y = \langle b, a\rangle$. Thus

$$\begin{aligned} \langle x, x\rangle &= \langle x, x^{-1}\rangle.\langle 1, x^2\rangle \\ &= (y^{-1}\langle x, 1\rangle^{-1} y \langle x, 1\rangle)\langle 1, x\rangle^2 \end{aligned}$$

so $\langle x, x\rangle$ is a product of a commutator and a square, as desired.

5 Set-theoretic analogues of model-theoretic notions

We have seen above how appropriate model-theoretic constructions may correspond to constructions for independences between weak versions of the axiom of choice. I now consider what has for model theory become a very significant notion, since the Baldwin–Lachlan Theorem (Baldwin and Lachlan, 1971), and which has given a lot of impetus to the study of models of small Morley rank, namely the notion of a strongly minimal set. The corresponding notion in set theory already existed, though the name of 'amorphous' did not emerge until later. The definitions are as follows. A model is said to be *minimal* if it is infinite, but is not the disjoint union of two definable infinite sets, and *strongly* minimal means that this persists even under elementary extensions. The corresponding set-theoretic notion, (which is vacuous in the presence of the axiom of choice) is that a set is said to be *amorphous* if it is not the disjoint union of any two infinite sets (definable or not).

In Cherlin *et al.* (1985) an analysis of strongly minimal sets is given which involves consideration also of *strictly* minimal sets, being those which carry no non-trivial 0-definable equivalence relation. The idea is that general strongly minimal sets may be constructed from these by adjoining appropriate 'fibres'. Correspondingly there is a notion of *strictly* amorphous set, being one which carries no non-trivial partition at all.

Let us note that our definition of 'amorphous' is in truth analogous rather to a strengthening of 'strongly minimal' which could be called *higher-order* strongly minimal, since we are working in set theory, and no restriction is placed on the type of variable used in any definition. It is for this reason that it turns out that the only examples of amorphous sets are those which correspond to $\aleph_0$-*categorical* strongly minimal sets.

The aim of Truss (in press), only partially realized, is to give a 'classification' of amorphous sets. Now we have to be clear about what is meant by this. Usually, a classification theorem for a class of structures will isolate invariants ('classifiers') corresponding to the structures so that two structures receive the same classifier if and only if they are isomorphic. This is no good here, since it is easy, essentially by the construction given in Section 2, to construct Fraenkel–Mostowski models containing arbitrarily large sets (or even proper classes) of amorphous sets which look essentially indistinguishable set-theoretically, for example which are strictly amorphous. What is rather wanted is a notion of 'externally' isomorphic, or what comes down to the same thing, elementarily equivalent, inside a suitable structure. With this idea, it is argued in Truss (in press) that there is just a *set* of classifiers, which has cardinality $2^{\aleph_0}$. This is so far only proved with regard to certain special classifiers, sufficient to capture the 'bounded' amorphous sets.

An amorphous set U is said to be *bounded* if it has a strictly amorphous partition into finite sets. We gave an example of a strictly amorphous set in Section 2, and a modified construction produces a variety of bounded amorphous sets. One of the main results of Truss (in press) is that there is only one bounded amorphous set corresponding to each classifier. For the reasons given above, one has to be careful about what exactly is meant. The precise statement is an example of a 'reconstruction' result, which aims to characterize the properties of a model internally. Thus, a certain inner-model construction is performed, giving rise to a submodel $\mathfrak{N}$ of a model $\mathfrak{M}$. The desire is to show that in a sense, $\mathfrak{N}$ has not 'forgotten' where it came from. That is, there is a notion of forcing in $\mathfrak{N}$ such that on adjoining an $\mathfrak{N}$-generic filter $\mathscr{F}$, we return to $\mathfrak{M}$; or less ambitiously, when we perform the same construction by which $\mathfrak{N}$ was formed from $\mathfrak{M}$ inside $\mathfrak{N}[\mathscr{F}]$, we obtain $\mathfrak{N}$ again. A method like this was used in Truss (1974*b*) to obtain (admittedly a very weak) reconstruction result for Solovay's model (Solovay, 1970). In fact it seems true to say that it is generally the case that very strong hypotheses have to be imposed on $\mathfrak{N}$ in order to make the reconstruction work.

The main subdivision of the class of all amorphous sets is as follows.

(i) U is bounded (as just defined).

(ii) U is said to be of *projective type* if there is some non-degenerate pregeometry (as defined by Cameron (1990) for instance) on U satisfying certain conditions (such as the exchange property). Observe that from this it will follow that U is unbounded. This case splits into two, depending on whether there is or is not a bound on the cardinalities of the finite fields associated with geometries on (partitions of) U.

(iii) U may be unbounded but not of projective type.

Theorem 3.3 (i), (ii) illustrated two of these cases. Here is another.

THEOREM 5.1. There is an FM model in which the set of atoms forms an unbounded amorphous set which is not of projective type.

Proof. We give just one possible construction. Let $\mathfrak{M}$ be a model for FMC in which the set U of atoms has cardinality $\aleph_0$ and let $U = \{u_n : n \in \omega\}$. Let G be the group of all permutations of U with finite support, and for each $k \geqslant 0$ let π_k be the partition

$$\{\{u_0, u_1, \ldots, u_{2^k-1}\}, \{u_{2^k}, u_{2^k+1}, \ldots, u_{2.2^k-1}\},$$
$$\{u_{2.2^k}, u_{2.2^k+1}, \ldots, u_{3.2^k-1}\}, \ldots\}$$

of U. Let G_k be the setwise stabilizer of π_k in G, and let $\mathscr{F}$ be the filter generated by $\{G_k : k \geqslant 0\} \cup \{G_u : u \in U\}$. Then $\mathscr{F}$ is closed under conjugacy, since G contained only elements of *finite* support (and this was why we had to restrict to that subgroup), so we obtain a corresponding

FM model $\mathfrak{N}$. To see that U is amorphous in $\mathfrak{N}$ note that for any k, $\bigcap\{G_i : i \leqslant k\} \cap \bigcap\{G_{u_j} : j < 2^k\}$ acts transitively on $\{u_i : i \geqslant 2^k\}$, and the arguments previously used apply. Clearly U has partitions into 2^k-element sets for each k, so U is unbounded amorphous in $\mathfrak{N}$. □

The proof of the following result is given in Truss (in press).

THEOREM 5.2. Suppose that the set U of atoms is strictly amorphous, and that $W \subseteq V(U)$ is a transitive model of FM containing U for which the standard sets fulfil AC. Then W is equal to the Fraenkel subuniverse of $W[w]$, where w is a W-generic well-ordering of U in type ω.

Here by a 'W-generic well-ordering in type ω' we mean a well-ordering obtained in the natural way from a W-generic subset of the notion of forcing P in W consisting of all finite sequences of distinct elements of U, partially ordered by end-extension. Similar work is carried out by Creed (to appear) with regard to the notion of an o-minimal set. According to the definition of Pillay and Steinhorn (1986), a structure $\mathfrak{A}$ is *o-minimal* if it is linearly ordered, and the only definable subsets of $\mathfrak{A}$ (with parameters allowed) are finite unions of intervals with endpoints in $A \cup \{\pm\infty\}$. The analogous definition of U being *o-amorphous* is that U is linearly ordered, and the *only* subsets of U are finite unions of intervals with endpoints in $U \cup \{\pm\infty\}$.

What is the correct notion of 'strictly o-amorphous'? Note that any o-amorphous set can automatically be split into arbitrarily many infinite subsets, just by taking intervals. So it may seem as if something a bit more complicated is required, and one possible definition is as follows: $(X, <)$ is said to be *strictly o-amorphous* if it is o-amorphous, and there is no disjoint pair (a, b), (c, d) of isomorphic or anti-isomorphic non-empty intervals. This however is easily seen to be equivalent to saying that there is no partition of U containing infinitely many non-singleton finite sets. One of the main results of Creed (to appear) is then that there is essentially only one strictly o-amorphous set, subject to the same provisos as Theorem 5.2.

What about an analogue of Theorem 5.1? The notion of a *bounded* o-amorphous set may be introduced much as before, but now things work out quite differently.

THEOREM 5.3. (Creed) Every o-amorphous set is bounded.

The reason for this is roughly speaking that any non-trivial partition of U into finite sets provides us with a partition into finitely many intervals, and one of these can be canonically selected. If U were unbounded o-amorphous, we would then be able to select a nested ω-sequence of intervals, which is clearly impossible in an o-amorphous set.

Finally I mention that if in the definition of U being o-amorphous we relax the requirement that the endpoints of the intervals should *lie* in U, then the class of sets thus obtained is greatly enriched, and many interesting configurations are possible. It is unclear at present however whether it will be feasible to carry out any sort of 'classification' in this case too.