COMPUTER SCIENCE TRIPOS 1A 2013
Paper 2 Question 6

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Let $R \subseteq U \times U$ be a relation on a set U.

Part a

Let $R^\dagger$ be defined by

$$\frac{}{\langle a, b\rangle}\ \langle a, b\rangle \in R \qquad\qquad \frac{\langle a, b\rangle\ \ \langle b, c\rangle}{\langle a, c\rangle}$$

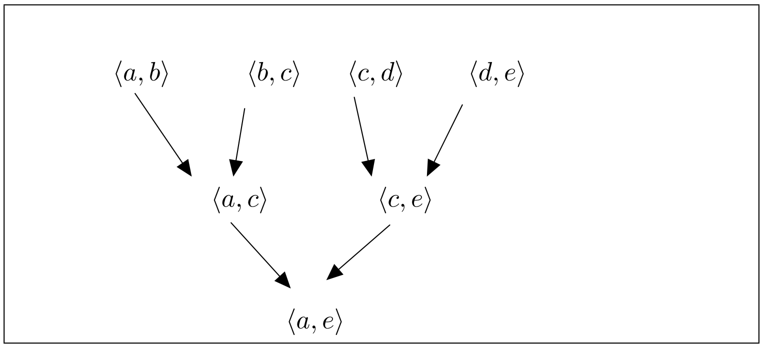
and $R^\bullet$ be defined by

$$\frac{}{\langle a, b\rangle}\ \langle a, b\rangle \in R \qquad\qquad \frac{\langle b, c\rangle}{\langle a, c\rangle}\ \langle a, b\rangle \in R$$

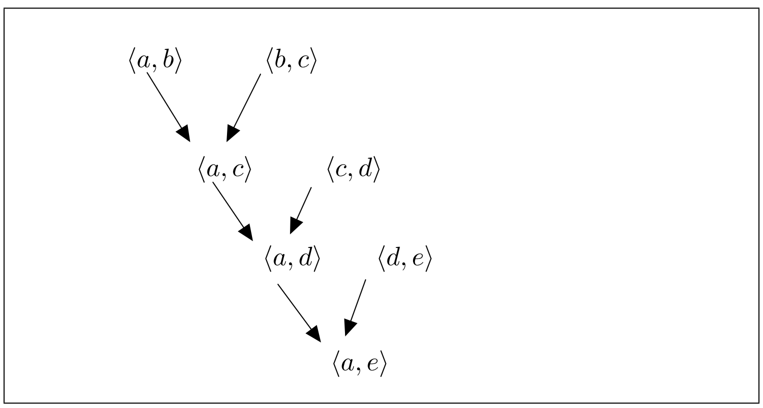
(I am writing ordered pairs with angle brackets. Always have done; too old to change)

You have to decide which of $R^\dagger$ and $R^\bullet$ are included in the other (or none of the above). Now clearly both are supersets of R, and the only difference in the two sets of rules is in the rules on the right. The rule for $R^\dagger$ is more permissive in the sense that for $R^\dagger$ it is sufficient for $\langle a, b\rangle$ to be in $R^\dagger$ whereas in the corresponding rule for $R^\bullet$ we require $\langle a, b\rangle \in R$, which is a stronger condition. Thus more stuff gets put into $R^\dagger$, so we must have $R^\bullet \subseteq R^\dagger$. Is that enough of an explanation? Possibly not, and I might return to it and do a bit of induction…but *later*.

Now my instinct is that it's perfectly obvious that $R^\dagger = R^\bullet$. Consider these two derivations:



Part of an evil scheme to show that $\langle a, e\rangle \in R^\dagger$.



Part of an evil scheme to show that $\langle a, e\rangle \in R^\bullet$.

In both pictures the ordered pairs at the top of the branches are of course in R. This does rather suggest that $R^\dagger = R^\bullet$, but how are we to prove it?

This reminds me of a standard example, which i will now discuss, since it is actually a slightly simpler case.

Finite sets defined by rule-induction

Let us say a set is finite[1] if it is either the empty set $\emptyset$ or is $X \cup \{y\}$ where X is finite. In our syntax this is

$$\frac{}{\emptyset} \qquad\qquad \frac{X}{X \cup \{y\}}$$

You could put a side-condition "$y \in U$" if it was finite subsets of U that you were generating.

[1] This is a standard definition of finite, originally due to Kuratowski

Now we claim that, if X and Y are both finite, so is $X \cup Y$. Is this obvious? It might be. However the striking thing as that, although the expression '$X \cup Y$' is symmetric in 'X' and 'Y', the proof (that $X \cup Y$ is finite if X and Y are) is *not* symmetric in 'X' and 'Y'. How does it go? Well, think about why it's true. For any X, it's true if Y is the empty set... so we do it by induction on 'Y'. Suppose $Y = Y' \cup \{y\}$ and that $X \cup Y'$ is finite. Then $X \cup (Y' \cup \{y\}) = (X \cup Y') \cup \{y\}$ and $X \cup Y'$ is finite by induction hypothesis, whence $(X \cup Y') \cup \{y\}$ is finite too. So what are we proving by induction?

We are proving by induction on Y that: for all X, if X is finite, then $X \cup Y$ is finite. This property (of Y, a formula with 'Y' free)

$$(\forall X)(Fin(X) \to Fin(X \cup Y))$$

is possessed by the empty set. Also, if it is possessed by Y, it is possessed by $Y \cup \{y\}$. So, by ("rule") induction, it is possessed by all finite Y.

Now to return to the challenge of showing that $R^\dagger \subseteq R^\bullet$.

Remember, when you are trying to show that a set A is a subset of a set B, this means you have to prove that every member of A is a member of B. In this case A is $R^\dagger$ which is an inductively defined set and supports an induction principle. So the obvious thing to try is to prove by $R^\dagger$-induction that everything in $R^\dagger$ is in $R^\bullet$.

The base case: We put $\langle a, b\rangle$ into both $R^\bullet$ and $R^\dagger$ as long as $\langle a, b\rangle \in R$. For the induction step we have to establish that if $\langle a, b\rangle$ and $\langle b, c\rangle$ are both in $R^\bullet$ then so is $\langle a, c\rangle$.

Clearly we are going to have to do this by an $R^\bullet$-induction. By analogy with the finite sets case it looks as if we should be trying to prove by $R^\bullet$-induction that

$$(\forall a)(\langle a, b\rangle \in R^\bullet \to \langle a, c\rangle \in R^\bullet)$$

holds for all $\langle b, c\rangle \in R^\bullet$

...or perhaps trying to prove by $R^\bullet$-induction that

$$(\forall c)(\langle b, c\rangle \in R^\bullet \to \langle a, c\rangle \in R^\bullet)$$

holds for all $\langle a, b\rangle \in R^\bullet$.

Which do we do? To answer this we have to look at the induction rule for $R^\bullet$, where the two ordered pairs have different conditions on them. If we were to try to prove the first we would have to first show (the base case) that if $\langle a, b\rangle \in R^\bullet$ and $\langle b, c\rangle \in R$ then $\langle a, c\rangle \in R^\bullet$. But that's the wrong way round, so let's try the second. To prove the second we would have to first show (the base case) that if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R^\bullet$ then $\langle a, c\rangle \in R^\bullet$. But that is given.

For the induction step suppose

$$(\forall c)(\langle b, c\rangle \in R^\bullet \to \langle a, c\rangle \in R^\bullet) \tag{1}$$

and that $\langle x, a\rangle \in R$. We want to infer

$$(\forall c)(\langle b, c\rangle \in R^{\bullet} \to \langle x, c\rangle \in R^{\bullet}) \tag{2}$$

And this we do as follows. Let c be arbitrary, and suppose $\langle b, c\rangle \in R^{\bullet}$; then $\langle a, c\rangle \in R^{\bullet}$ by induction hypothesis, (1) above. But if $\langle a, c\rangle \in R^{\bullet}$ and $\langle x, a\rangle \in R$ then $\langle x, c\rangle \in R^{\bullet}$ by the right-hand rule for $R^{\bullet}$. Then we infer (2) by universal generalisation, since c was arbitrary.

Part b

Let $R^{\diamond}$ be defined by

$$\frac{}{\langle a, b\rangle}\ \langle a, b\rangle \in R \qquad\qquad \frac{\langle b, c\rangle}{\langle a, d\rangle}\quad \langle a, b\rangle, \langle c, d\rangle \in R$$

(i)

Prove or disprove

$$R^{\diamond} \subseteq \bigcup_{n \in \mathbb{N}} R^{2n+1}$$

When you think about what the $\diamond$-rules do, it's pretty obvious that $R^{\diamond}$ is going to be the union of all the odd powers of R, so we should be aiming to prove this allegation, and the obvious way to prove it is by $R^{\diamond}$-induction.

The induction is quite easy. Suppose $\langle x, y\rangle \in R^{\diamond}$ and also in $\bigcup_{n \in \mathbb{N}} R^{2n+1}$. Suppose, too, that $\langle v, x\rangle$ and $\langle y, z\rangle$ are both in R. Then $\langle v, z\rangle$ is in $R^{\diamond}$ by the $\diamond$ rules, and is also in $\bigcup_{n \in \mathbb{N}} R^{2n+1}$.

(ii)

This looks true too. How are you going to prove this inclusion? The thing on the left of the '$\subseteq$' is a union of countably many things, so we could do it by proving that each of those countably many things is included in the RHS, and it's pretty obvious that you want to do that by induction on 'n'.

(iii)

$$(R^{\diamond})^{-1} = (R^{-1})^{\diamond}.$$

This is obviously true, but I can't see offhand the right way to make this fact obvious. Perhaps we should use the hint: *we are allowed to assume that, for each $n \in \mathbb{N}$, the relation $R^n \subseteq U \times U$ obeys $R \cdot R^n = R^{n+1} = R^n \cdot R$, and obeys $(R^n)^{-1} = R^{-n}$.*