COMPUTER SCIENCE TRIPOS Part IA 2013 Paper 2 Q 5

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Part (a)

This is quite a clever question. An aside on notation: writing '$[n]$' for $\{m \in \mathbb{N} : m < n\}$ is not standard, tho' it is not too far from standard. More usual is '$[0, n]$', as an instance of '$[m, n]$' meaning $\{i \in \mathbb{N} : m \leq i \leq n\}$, but one can see why the question-setter would strip this notation down... the 'm' variable is not being called upon to do anything in this context.

Part (i)

I'd be happier if they'd asked you to *implement* the disjoint union rather than define it (beco's it's actually implementation that is going on here rather than definition) but this deplorable terminology is now entrenched. (you define disjoint union operationally by describing what it does, by giving a spec; you implement it in such a way that it fits the spec). The point is still worth making nevertheless.

Let us define(!) $[m] \sqcup [n]$ to be $[m] \times \{0\} \cup [n] \times \{1\}$. (I am using the $\sqcup$ notation rather then the $\cup$ with a dot beco's i don't seem to have a LaTeX code for the symbol the question-setters are using, and also beco's $\sqcup$ is pretty standard and you should expect to see it (which is why LaTeX has a symbol for it: `\sqcup`). Then we can biject $[m] \sqcup [n]$ with $[m + n]$ by sending the pair $\langle i, j \rangle$ to $m \cdot j + i$. Apparently at this stage you're supposed to say something about why this is an injection/bijection. I can't think of anything to say; if you can't see that it's a bijection then you haven't been paying attention. If you can think of anything intelligent to put in here, Dear Reader, please supply it and you will be rewarded with the usual bribes.

Part (ii)

The bijection we want takes a pair $\langle i, j \rangle \in [m] \times [n]$ and sends it to $m \cdot j + i$.

Part (iii)

Old lags know that, for any set X, a subset of X can be thought of as a function from X to $\{0, 1\}$. More specifically: if X comes naturally equipped with a total order (as here) it is natural to think of a subset of X as a bit string of the same length as X, with the 0s and 1s telling which elements of X to put in and which to leave out. So if X is a subset of $[m]$ we think of it as a function from $[m]$ to $\{0, 1\}$, which is to say, a bit string. But compscis know how to turn a bit-string into a number: think of it as a binary representation of a natural number! To write it out formally...

$$X \mapsto \sum_{i \in X} 2^i$$

Part (iv)

You want to take a function from the naturals below m to the numbers below n, and somehow recover a number below n^m. There is only one sensible answer, but i can't see a good way of making this sensible answer obvious. The best i can suggest is that one compares part (iv) with part (iii). The difference between these two problems is basically that in part (iv) '2' has been replaced by 'n'. If you think about it for a bit you will come up with the idea that perhaps a function as in part (iv) can be thought of as something that gives you a base n expansion of some number, with m digits, in the sense that $f(i)$ (which is a number $< n$) can be the (number in the) ith position in the base-n expansion. Once you spot *that* it is easy to see that the fs collectively fill in the m different places in all possible ways.

Part (b)

> *Suppose there is a surjection $f : D \twoheadrightarrow (D \to D)$. Show that this happens if and only if D has precisely one element.*

If D has one element then $D \to D$ is the singleton of the identity function $\mathbb{1}_D$ and both D and $D \to D$ are singletons so there is a surjection as desired.

If D is empty then $D \to D$ is the singleton of the empty function. There can be no surjection from the empty set to a nonempty set, so again, we get the result we want.

Now suppose D is a set with at least two members. Let us name two of them 'a' and 'b'. Suppose further that $f : D \to (D \to D)$. We will show that f is not surjective.

The challenge is to cook up a function $\delta : D \to D$ which is not in the range of f. And we have to cook up such a function using only f, a and b. ...

Observe that we *have* to use *both* a and b. After all, we saw above that if f has only one member there *is* a surjection. We should expect a diagonal construction to appear, so tinkering with $f(x)$ applied to x would be a good thing to start with. And of course we have to alter the thing on the diagonal, so something like the following would be worth trying.

Define a function $\delta : D \to D$ by

$$\texttt{if } (f(x))(x) = a \texttt{ then } b \texttt{ else } a$$

The chief effect of this definition is that

$$(\forall d \in D)(\delta(d) \neq (f(d))(d)) \tag{1}$$

We now claim that δ is not in the range of f. For suppose δ were $f(d_0)$; we obtain a contradiction by considering $\delta(d_o)$.

$\underline{\delta}(d_o) = \underline{f(d_0)}(d_0)$ (the underlined parts are identical by definition).

But we also have

$\delta(d_o) \neq (f(d_0))(d_0)$ by (1)

giving us the contradiction we sought.

Observe that we used only the fact that D has two distinct elements. We had not assumed that D was finite. You can try [tho' you shouldn't] to prove this result by induction on the size of D, but that only proves it for D that are finite, and the angels will weep for you.

For the *cognoscenti*... we have also used excluded middle on $x = a$, in the definition of δ.

This question was clearly set by a theorist. I have recently been shown

Andrej Bauer: *On fixed-point theorems in synthetic computability*
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...which is a beautiful exposition of the ideas in this question.