Computer Science Tripos 1996 Paper 1 Question 8

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The recurrence

$$R: \ w(n,k) = w(n-2^k,k) + w(n,k-1)$$

can be justified as follows. Every representation of n pfatz as a pile of coins of size no more than 2^k pfatz either contains a 2^k pfatz piece or it doesn't. Clearly there are $w(n,k-1)$ representations of n pfatz as a pile of coins of size no more than 2^{k-1} pfatz so that's where the $w(n,k-1)$ comes from. The other figure arises from the fact that a representation of n pfatz as a pile of coins of size no more than 2^k pfatz and containing a 2^k pfatz piece arises from a representation of $n-2^k$ pfatz as a pile of coins of size no more than 2^k.

Base case. $w(n,0) = 1$. That should be enough.

To derive $w(4n,2) = (n+1)^2$, substitute $4n$ for n, and 2 for k in R, getting

$$w(4n,2) = w(4n-2^2,2) + w(4n,1)$$

But this rearranges to

$$w(4n,2) = w(4(n-1),2) + w(4n,1)$$

$w(4n,1)$ is $2n+1$, since we can have between 0 and $2n$ 2-pfatz pieces in a representation of $4n$. This gives

$$w(4n,2) = w(4(n-1),2) + 2n + 1$$

This is a bit clearer if we write this as $f(n) = f(n-1)+2n+1$. This recurrence relation obviously gives $f(n) = (n+1)^2$ as desired .

We can always get an estimate of $w(n,k)$ by applying equation R recursing on n, and this works out quite nicely if n is a multiple of 2^k because then we hit 0 exactly, after $n/(2^k)$ steps. Each time we call the recursion we add $w(n,k-1)$ (or rather $w(n-y,k-1)$ for various y) and clearly $w(n,k-1)$ is the biggest of them. So $w(n,k)$ is no more than $n/(2^k) \cdot w(n,k-1)$.

Finally, using R with 2^{k+1} for n again we get $w(2^{k+1},k) = w(2^k,k) + w(2^{k+1},k-1)$. The hint reminds us that every representation of 2^k pfatz using the first k coins gives rise to a representation of 2^{k+1} pfatz using the first $k+1$ coins. Simply double the size of every coin. It's also true that every representation of 2^k pfatz using the first k coins gives rise to a representation of 2^{k+1} pfatz using the first $k+1$ coins by just adding a 2^k pfatz piece. The moral is: $w(2^{k+1}, k+1) = 2 \cdot w(2^k, k)$. This enables us to prove the left-hand inequality by induction on k.

To prove the right-hand inequality we note that any manifestation of 2^k pfatz using smaller coins can be tho'rt of as a list of length k where the ith member of the list tells us how many 2^i pfatz coins we are using. How many lists of length k each of whose entries are at most 2^k are there? Answer $(2^k)^k$, which is 2^{k^2}.