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YABLO'S PARADOX AS A THEOREM OF MODAL LOGIC

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ABSTRACT

We (further) demystify Yablo's paradox by showing that it can be thought of as the fact that the formula $\Box(p \leftrightarrow \Box\neg p)$ is unsatisfiable in the modal logic KD4 characterised by frames that are strict partial orders without maximal elements. This modal treatment also unifies the two versions of Yablo's paradox, the original version and its dual.

1. Introduction

This paper is meant as a follow-up to [For04]. We reinforce the message from that paper that Yablo's striking puzzle is really — despite the name — no paradox at all. In corollary 1 below we show that if we think of the infinite family of propositions in the puzzle as possible worlds then the moral is that Yablo's paradox is simply a theorem in a particular modal logic.

Initially, before invoking any modal machinery, we consider Yablo's paradox [For04] in the form:

$$(\forall i)\,[P(i) \leftrightarrow (\forall j > i)\neg P(j)\,] \tag{A}$$

We will deduce a contradiction from (A), and make a note of what assumptions we had to make to secure that outcome.

Proof. Let i be arbitrary. We will prove both $P(i)$ and $\neg P(i)$.

Assume $P(i)$, and let $j > i$ be arbitrary. Then $\neg P(j)$. Then $(\exists k > j)\,P(k)$. But $k > i$ by transitivity of $>$, and $P(k)$ contradicts $(\forall j > i)\neg P(j)$ and hence contradicts $P(i)$. So $\neg P(i)$.

Now assume $\neg P(i)$. Then there is $j > i$ with $P(j)$. But in the preceding paragraph we refuted $P(j)$ for arbitrary $j > i$, so this case is impossible too, whence $P(i)$.

$\Box$

The only assumptions we have used are as follows:

1. $>$ is transitive
2. $>$ is irreflexive
3. $(\forall i)\ (\exists j)\ (j > i)$.

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That is to say: the minimal nonlogical assumption needed to obtain a contradiction from (A) is that $>$ should be a strict partial order with no maximal element. Condition 3 above, is sometimes called *seriality* by modal logicians.

Introducing a modal logic allows for an object-level and finitary proof of the unsatisfiability of the formula labelled A.

2. The Propositional Modal Logic KD4

The thought now is that we can make progress by thinking of the $P(i)$ in the statement of Yablo's paradox not as an infinite family of atomic propositions but as a single proposition evaluated in lots of worlds in a Kripke model. Thus the derivability of Yablo's paradox should be the same fact as the theoremhood of a particular formula in the normal modal logic characterised by frames whose accessibility relation satisfies 1-3 above.

That logic is normal modal logic KD4 where K is $\Box(p \to q) \to (\Box p \to \Box q)$, D is $\Box p \to \Diamond p$ and 4 is $\Box p \to \Box\Box p$. The axiom D characterises seriality $\forall x.\exists y.\ R(x, y)$ and the axiom 4 characterises transitivity $\forall x, y, z.\ R(x, y) \& R(y, z) \Rightarrow R(x, z)$. Although irreflexivity is not actually characterised by any of the axioms, it is known that the class of irreflexive, transitive and serial frames also characterises this logic [PB02].

Now what, exactly, is the formula we are trying to prove? Our first thought was that we should be able to refute $p \leftrightarrow \Box\neg p$, and thereby obtain a proof of

$$\neg(p \leftrightarrow \Box\neg p) \tag{F}$$

However, this cannot be right, for formula (F) easily implies $p \leftrightarrow \Box p$, and this is certainly not a theorem of KD4.

Consider now the standard translation [PB02] of propositional modal logic into first-order logic given below with one free variable x:

$$\begin{array}{lcl}
ST(x, p) & = & p(x) \\
ST(x, \varphi \wedge \psi) & = & ST(x,\varphi) \wedge ST(x,\psi) \\
ST(x, \varphi \vee \psi) & = & ST(x,\varphi) \vee ST(x,\psi) \\
ST(x, \varphi \to \psi) & = & ST(x,\varphi) \to ST(x,\psi) \\
ST(x, \Box\varphi) & = & \forall y.R(x,y) \to ST(y,\varphi) \\
ST(x, \Diamond\varphi) & = & \exists y.R(x,y) \wedge ST(y,\varphi)
\end{array}$$

Replacing $R(x,y)$ by $y > x$ and replacing $\forall y.y > x \to \varphi$ by $\forall y > x.\varphi$ gives:

$$\begin{aligned}
ST(0,\ \Box(p \leftrightarrow \Box\neg p)) &= \forall y.R(0,\ y) \to (p(y) \leftrightarrow \forall z.R(y,\ z) \to \neg p(z)) \\
&= \forall y > 0.(p(y) \leftrightarrow \forall z > y.\neg p(z))
\end{aligned}$$

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}\ \mathrm{id} \qquad \frac{\Box\Gamma, \Gamma \vdash X}{\Box\Gamma, \Sigma \vdash \Box X, \Delta}\ \mathrm{KD4}\ \ X = \emptyset \text{ or } X = \{\varphi\}$$

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ \neg\mathrm{L} \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ \neg\mathrm{R}$$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ {\to}\mathrm{L} \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ {\to}\mathrm{R}$$

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ {\vee}\mathrm{L} \qquad \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ {\vee}\mathrm{R}$$

$$\frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ {\wedge}\mathrm{L} \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ {\wedge}\mathrm{R}$$

$$\frac{\Gamma, \varphi, \psi \vdash \Delta \quad \Gamma \vdash \varphi, \psi, \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ {\leftrightarrow}\mathrm{L} \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta \quad \Gamma, \psi \vdash \varphi, \Delta}{\Gamma \vdash \varphi \leftrightarrow \psi, \Delta}\ {\leftrightarrow}\mathrm{R}$$

Figure 1: Cut-free complete sequent calculus for the modal logic KD4.

which is the essence of A. Thus the modal formula we want is the formula Y below and this *is* a theorem of KD4, as we will now show:

$$\neg\Box(p \leftrightarrow \Box\neg p) \qquad \text{(Y)}$$

3. A Sequent Calculus for KD4

There are now two ways to proceed. One is to argue semantically that formula Y is valid, for example, by arguing semantically that its negation is unsatisfiable. The other is to give a syntactic proof of Y in an appropriate proof calculus for the modal logic KD4.

When we take into account the previous work on this topic [For04], the route via a proof calculus has two virtues: (i) the reasoning is at the object-level, and (ii) the proof is finitary. We therefore follow the syntactic route.

The sequent calculus rules for the logic KD4 are shown in Figure 1 where each of Γ and Δ and Σ are finite sets of formulae and X is a set containing at most one formula. The only odd rule is the rule KD4, which is applicable to any $\Box$-formula on the right-hand-side of the sequent, but which is also applicable if there are no such $\Box$-formulae. The rule has similarities to a modal rule, often called (T), that captures reflexivity, but reflexivity is too strong since KD-frames need not be reflexive.

The easiest way to read the KD4 rule is to see it as a move from a world w (the conclusion) to an R-accessible v (the premise). For suppose that w makes all the formulae in $\Box\Gamma$, Σ true and makes all the formulae in $\Box X$, Δ false. If $X = \{\varphi\}$ then w must have an R-successor v which makes Γ true and φ false since $\Box\varphi$ false at w is the same as $\Diamond\neg\varphi$ true at w. Else if $X = \emptyset$, then again some R-successor v must exist by seriality of R. By transitivity, v must also make $\Box\Gamma$ true. But it need not make the formulae in Σ true nor make the formulae in Δ false, which is why they both "disappear" in passing from the conclusion w to the premise v This is essentially the soundness of the KD4 rule with respect to KD4-frames.

More generally, the sequent calculus is sound and cut-free complete for the logic KD4 in the following sense [Gor99].

Theorem 1. *A formula φ is KD4-valid iff there is a derivation of the sequent $\vdash \varphi$ in the sequent calculus from Figure 1.*

Proposition 1. *The rule $\leftrightarrow$L shown in Figure 1 is derivable by defining $\varphi \leftrightarrow \psi$ as $(\psi \to \varphi) \wedge (\varphi \to \psi)$ as shown below:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \varphi, \psi \vdash \Delta \qquad \dfrac{}{\Gamma, \varphi \vdash \varphi, \Delta}\,\text{id}}{\Gamma, \varphi, \varphi \to \psi \vdash \Delta}\,{\to}\text{L} \qquad \dfrac{\dfrac{}{\Gamma, \psi \vdash \psi, \Delta}\,\text{id} \qquad \Gamma \vdash \varphi, \psi, \Delta}{\Gamma, \varphi \to \psi \vdash \psi, \Delta}\,{\to}\text{L}}{\Gamma, \psi \to \varphi, \varphi \to \psi \vdash \Delta}\,{\to}\text{L}}{\Gamma, (\psi \to \varphi) \wedge (\varphi \to \psi) \vdash \Delta}\,\wedge\text{L}}{\Gamma, \varphi \leftrightarrow \psi \vdash \Delta}\,\text{defn}
$$

4. Formalising Yablo's Paradox in KD4

The proof object is a bit too large for comfort, so we start in a small way by proving a fragment of it that has more than one occurrence in the main proof.

Proposition 2. *The sequent $p, \Box\neg p, \Box(p \leftrightarrow \Box\neg p)\neg$ is derivable in KD4:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\neg p, p, \Box(p \leftrightarrow \Box\neg p) \vdash^6 p}\,\text{id} \qquad \dfrac{\dfrac{\dfrac{}{p, \Box\neg p, p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash p}\,\text{id}}{\neg p, \Box\neg p, p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash \neg p}\,\neg\text{R};\neg\text{L}}{\Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^7 p, \Box\neg p}\,\text{KD4}}{\Box\neg p, p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^5 p}\,\leftrightarrow\text{L}}{\neg p, \Box\neg p, p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^4}\,\neg\text{L}}{p, \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^3}\,\text{KD4}
$$

Proposition 3. *The formula $\neg\Box(p \leftrightarrow \Box\neg p)$ is KD4-valid.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Proposition 2}}{p, \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^{11}} \qquad \dfrac{}{p, \Box(p \leftrightarrow \Box\neg p) \vdash^{12} p, \Box\neg p}\,\text{id}}{p, p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^{10}}\,{\leftrightarrow}\text{L}}{p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^{9} \neg p}\,\neg\text{R}}{\Box(p \leftrightarrow \Box\neg p) \vdash^{8} p, \Box\neg p}\,\text{KD4}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Proposition 2}}{p, \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^{3}} \qquad \dfrac{\text{See above}}{\Box(p \leftrightarrow \Box\neg p) \vdash^{8} p, \Box\neg p}}{p \leftrightarrow \Box\neg p, \Box(p \leftrightarrow \Box\neg p) \vdash^{2}}\,{\leftrightarrow}\text{L}}{\Box(p \leftrightarrow \Box\neg p) \vdash^{1}}\,\text{KD4}}{\vdash^{0} \neg\Box(p \leftrightarrow \Box\neg p)}\,\neg\text{R}$$

We will explain the in-line numerical annotations on $\vdash$ in Section 5.

Corollary 1. *No modal formula obeys Yablo's Paradox.*

5. Reading off the Proof from the Derivation

Here is an attempt to read off the natural language proof of Yablo's Paradox from the derivation of the formula $\neg\Box(p \leftrightarrow \Box\neg p)$.

We view the derivation upwards, from the end-sequent up to the leaves, as a proof by contradiction. Each sequent $\Gamma \vdash \Delta$ in the derivation is given the following semantic reading: if every member of Γ is true then every member of Δ is false. Thus the (id) leaves are all obviously contradictions of the form: if all members of the left hand side (including p) are true then all the members of the right hand side (including p) are false.

Now consider the sequents, numbered by the superscript on their turnstile and look at the derivation of Proposition 2 for the sequents numbered 4-7:

0. Suppose that $\neg Y$ is false
1. Then Y is true
2. Then $p \leftrightarrow \Box\neg p$ must be true at $i = 0$
3. So suppose p is true and $\Box\neg p$ is true at $i = 0$
4. Pick any $j > 0$. It must make $\neg p$ and $\Box\neg p$ and $p \leftrightarrow \Box\neg p$ true
5. That is, $j > 0$ must make p false and $\Box\neg p$ and $p \leftrightarrow \Box\neg p$ true
6. One way for $j > 0$ to make $p \leftrightarrow \Box\neg p$ true is to make p true and $\Box\neg p$ true. But this gives a contradiction since it makes p both true and false
7. Else $j > 0$ makes both p and $\Box\neg p$ false, which again gives a contradiction since it already must make $\Box\neg p$ true

8. Else suppose that p is false and $\Box\neg p$ is false at $i = 0$
9. Pick any $j > 0$. It must make $\neg p$ false and $p \leftrightarrow \Box\neg p$ true
10. That is, $j > 0$ must make p true and $p \leftrightarrow \Box\neg p$ true
11. If $j > 0$ makes both p and $\Box\neg p$ true then we have a contradiction via Proposition 2 as before
12. Else if $j > 0$ makes both p and $\Box\neg p$ false then we again have a contradiction since it already makes p true.

6. A dual version

There is also a dual version of Yablo's paradox, although not much is usually made of this fact:

$$(\forall i)[P(i) \leftrightarrow (\exists j > i)\neg P(j)] \tag{A*}$$

We can deduce a contradiction from (A*) too.

Proof. Let i be arbitrary.

Suppose $\neg P(i)$. Then $\forall j > i.\ P(j)$. Let j be an arbitrary $j > i$. Then we have $P(j)$, which implies that $\exists k > j.\ \neg P(k)$. But this k must satisfy $P(k)$ since $k > j > i$ and $\neg P(i)$. Contradiction: hence $P(i)$.

So $P(i)$ holds. Hence there is $j > i$ such that $\neg P(j)$. But we have already shown that $\neg P(j)$ for arbitrary j in the previous paragraph.

□

The modal analysis dual to that for the original version (A) of Yablo's paradox will throw up a corresponding modal principle $\neg\Box(p \leftrightarrow \Diamond\neg p)$. Pleasingly this turns out to be the same modal principle as $\neg\Box(p \leftrightarrow \Box\neg p)$, and we need no more than the duality of $\Box$ and $\Diamond$, captured by $\Box p \leftrightarrow \neg\Diamond\neg p$, to establish this equivalence:

$$\neg\Box(p \leftrightarrow \Diamond\neg p) \tag{1}$$

$$\leftrightarrow \neg\Box(\neg p \leftrightarrow \neg\Diamond\neg p) \tag{2}$$

$$\leftrightarrow \neg\Box(\neg p \leftrightarrow \Box p) \tag{3}$$

$$\text{i.e. } \neg\Box(q \leftrightarrow \Box\neg q) \tag{4}$$

7. Conclusion

The debate over Yablo's puzzle has taken its cue from Yablo's original motivation of studying the rôle of self-reference. This preoccupation with a narrow syntactic question has tended to distract attention from the question — which we think is more interesting — of what this very interesting paradox actually *means*.

References

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